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ROBUST CONTROL FOR LINEAR SYSTEMS WITH STRUCTURED UNCERTAINTY

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The research is focussed on the robust stabilization of linear systems with structured uncertainty. Analysis procedures and design techniques are provided from both the time domain (matrix) and frequency domain (transfer function) perspectives. The problem of combined (structured and unstructured) uncertainty is also addressed.

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SUMMARY

This final report summarizes the research carried out under the sponsorship of AFOSR (WPAFB) contract number F33615-86-K-3611, awarded to the University of Toledo for the period of June 1986 - June 1988. The main theme of the research is the robust stabilization of linear uncertain systems with structured uncertainty, from both the time domain and frequency domain perspectives. First, from an analysis point of view, two new results on improved measures of stability robustness are presented, one from the frequency domain (polynomial) perspective, the other from the state space (matrix theory) perspective. Then design algorithms are presented that completely utilize the structural information of the uncertainty. The frequency domain based design also considers a class of non-minimum phase systems. Finally the aspect of combined structured and unstructured uncertainty is addressed and design methods are presented that treat these two types of uncertainty in a unifying framework.

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FOREWORD

This report was prepared by the Department of Electrical Engineering, University of Toledo, Toledo, Ohio, under the Air Force contract F33615-86-K-3611. The work was performed under the direction of Dr. S. S. Banda, Captain Elkins, Lieutenant Timothy McQuade, and Lieutenant Mary Manning of the Air Force Flight Dynamics Laboratory, Wright-Patterson Air Force Base, Ohio.

The technical work was conducted by Dr. R. K. Yedavalli, Principal Investigator, and Dr. K. Wei, a post-doctoral research associate. The contract was performed during the period June 1986 - June 1988.

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I. INTRODUCTION AND PERSPECTIVE

It is well known that the inevitable presence of inaccuracies in the mathematical models of physical systems used for control design can severely compromise the resulting performance. These modeling uncertainties can be broadly categorized as (1) parametric (structured) uncertainty resulting from the variations in the parameters of the low frequency, finite dimensional model and (2) high frequency unstructured uncertainty resulting from the negligence of unmodeled dynamics. A "robust" control design is that design which maintains closed loop system stability and performance even in the presence of these perturbations.

The published literature on this "robust" stabilization and performance problem can be viewed from two perspectives, namely, (1) frequency domain viewpoint, and (2) time domain viewpoint. In the frequency domain viewpoint, the system is assumed to be described by transfer function representation, thereby making extensive use of "polynomial" theory, whereas the time domain treatment is naturally suited to systems represented by state space description, thereby making extensive use of "matrix" theory.

In the transfer function based frequency domain representation, the "structured" uncertainty is manifested in the form of variations in the coefficients of the numerator and denominator polynomials, whereas the "unstructured" uncertainty is reflected in terms of the specification of a norm bounded frequency dependent function. On the other hand, in the state space, time domain framework, "structured" uncertainty is manifested in the form of variations in the elements of the system matrices while "unstructured"

uncertainty takes the form of a norm bounded perturbation matrix as affecting the system.

In this research, we employ both frequency domain viewpoint as well as time domain viewpoint in treating the problem of stability robustness, with particular emphasis on "structured" uncertainty when treated alone. Towards this direction we present results on both "analysis" and on "design" techniques for linear systems with structured uncertainty. However, realizing that concentrating on only one type of uncertainty (either structured or unstructured) in the problem formulation is inadequate, since most of the applications of interest have both types of uncertainty present simultaneously, we also treat the combined uncertainty problem and present some interesting results on this aspect, again from both time domain and frequency domain viewpoints.

The report is organized as follows: Section II considers the analysis aspect, in which a brief review of "polynomial" and "matrix" approaches in the stability robustness analysis of linear systems is given. Then in subsections 2.2 and 2.3, two methods are presented, one for the polynomial case and the other for state space case, to improve the stability robustness bounds under dependent structured uncertainty. Section III addresses the synthesis aspect and presents a time domain control design method for robust stability and regulation in subsection 3.1, and in 3.2 a frequency domain based design method is given to simultaneously stabilize a class of non-minimum phase systems. Section IV considers the combined structured and unstructured uncertainty problem and presents both time domain based and frequency domain based design methods in subsections 4.1 and 4.2, respectively. Finally, Section V offers some concluding remarks and recommendations for future research.

II. ANALYSIS FOR STABILITY ROBUSTNESS OF LINEAR SYSTEMS WITH STRUCTURED UNCERTAINTY

In this section, we present some new results on the stability robustness analysis for linear systems with structured uncertainty described by both transfer function representation and state space representation. In the former case, the "structured" uncertainty is manifested in the form of variations in the coefficients of the numerator and denominator polynomials of the transfer function, while in the latter case, it is reflected in terms of the variations in the elements of the system matrices. Accordingly, the stability robustness analysis in transfer function formulation is addressed using the Hurwitz "polynomial" theory while that of the state space formulation is addressed using "matrix" theory. Thus we label the approaches for the former case as "polynomial" approaches and those for the latter case as "matrix" approaches. Also, in each of these perspectives, the robustness analysis problem can be either in the form of checking stability of the family of polynomials (or matrices) given the bounds on the parameters ("Hurwitz invariance tests") or in the form of determining the allowable bounds on the parameters to maintain stability ("stability robustness measures"). Before presenting the two new results, one from each perspective, in subsection 2.1, we briefly review the available literature on this aspect.

2.1 BRIEF REVIEW OF "POLYNOMIAL" AND "MATRIX" APPROACHES

Polynomial Approaches: In the framework of continuous time uncertain systems represented by transfer function polynomials, the robustness issue is cast as a problem of testing the Hurwitz invariance of a family of polynomials parametrized by an uncertain parameter vector $\underline{q} \in R^r$, which we denote as $f(s,q)$. The different methods available for this problem are dependent on the characterization of the "structure" of the uncertainty; i.e., the way the coefficients of the polynomial depend on the real parameters q_i $(i = 1,2,\ldots,r)$. This characterization can be divided into four categories:

Category P1. Independent variations: In this case, the coefficients vary independently of each other. For example,

$$f(s,\underline{q}) = s^3 + 3q_1s^2 + 4q_2s + 5, \qquad \underline{q}_i \le q_i \le \bar{q}_i, \qquad i = 1,2.$$

The family of polynomials generated in this category is called an "interval polynomial" family.

Category P2. Linear dependent variations: For this case, the coefficients vary linearly on the uncertain parameters q_i. For example,

$$f(s,\underline{q}) = s^5 + 3q_1s^4 + 4q_1s^3 + 2q_2s^2 + q_2s + 4q_3.$$

In addition, the type of polynomial

$$f(s,\underline{q}) = s^3 + (1 - q)s^2 + (3 - q)s + (3 - q)$$

is said to be "affinely linear dependent." The family of polynomials generated in this category is a "polytope" of polynomials.

Category P3. Multilinear case: In this case, the coefficients have product terms in the parameters but are such that they are linear functions of a single parameter when others are held constant. One example is

$$f(s,\underline{q}) = s^3 + 2q_1q_2s^2 + 4q_2q_3s + 5q_1q_2q_3$$

Category P4. Nonlinear case: The coefficients are nonlinear functions of the parameters. For example,

$$f(s,\underline{q}) = s^3 + 2q_1^3q_2s^2 + (4q_2^2 + 3q_3)s + 5$$

With this classification in mind, we can now present some results available for these cases. In [1], Guiver and Bose consider the "interval" polynomials and derive a maximal measure of robustness for quartics ($n \leq 4$, where n is the degree of the polynomial). Perhaps the best known and most significant result for the "interval" polynomial stability is the one by Kharitonov [2]. This result shows that of the 2^n extreme polynomials formed by restricting the coefficients to the endpoints of the allowable range of variation, stability can be determined by examining only four special members of this set (independent of the degree n). It may be noted that Kharitonov's result is a necessary and sufficient condition for the stability testing of interval polynomials but becomes a sufficient condition for the case of a polytope of polynomials. Kharitonov's result was introduced into the western literature by Barmish [3]. Later in [4], Bose recasts the Kharitonov analysis in a system theoretic setting, and in [5], Anderson, Jury, and Mansour present a refinement of Kharitonov's result by showing that fewer than four polynomials are required for stability testing for degree $n \leq 5$.

Some additional results on interval polynomials and some early results on the polytope of polynomials can be found in References [6] through [10].

The next important result for the "polytope" of polynomials is given in Bartlett, Hollot, and Lin [11] in which they provide necessary and sufficient conditions for checking stability. This result is now known as the "edge" theorem. It is in a way a Kharitonov-like result for the polytope of polynomials, in the sense that, instead of the vertices of a hypercube, one

has to check the "edges" of the polytope to ascertain stability. However, this result suffers from a "combinatorial explosion" problem. That is, the number of "edges" to check increases significantly with increase in the number of uncertain parameters. The most recent result for this problem, which circumvents somewhat the "explosion" problem, is given by Barmish [12]. Here a special function called "robust stability testing function," $H(\delta)'$, is constructed and robust stability is assured if and only if this function remains positive for all $\delta \geq 0$.

In subsection 2.2, we present a result which reduces the conservatism of the Kharitonov test (as a sufficient condition) in the testing of Hurwitz invariance of a family of polynomials with dependent coefficients. This dependency can belong to categories P2, P3, or P4.

<u>Matrix Approaches</u>: In the framework of continuous time uncertain systems described by state space representation, the robustness issue is cast as a problem of testing the stability of a family of real $n \times n$ matrices, parametrized by an uncertain parameter vector $\underline{q} \in R^r$, which we denote as $A(\underline{q})$. Specifically, one can write the uncertain system as

$$\dot{x}(t) = A(\underline{q})x(t) = [A_0 + E(\underline{q})]x(t)$$

where A_0 is an asymptotically stable matrix. The different methods available for this problem are dependent on the characterization of the perturbation matrix $E(\underline{q})$. If only a norm on the perturbation matrix E, $||E||$, is known, and bounds on it for stability of $A_0 + E$ are derived, then it is labeled as the "unstructured" uncertainty problem. Some of the results for this case are reported in References [13] through [21]. However, in this research we concentrate mainly on the "structured" uncertainty case. For this case, as in

the polynomial setting, the different methods available are again influenced by the characterization of the "structure" of the uncertainty; namely, the way the elements of E depend on the real parameters q_i $(i = 1,2,\ldots,r)$. Again, we can consider the following categories:

<u>Category M1. Independent variations</u>: In this case the elements of E vary independently, i.e.,

$$\underline{E}_{ij} \leq E_{ij} \leq \bar{E}_{ij} \quad \text{for all} \quad i,j = 1,2,\ldots,n$$

Another way of representing this situation is

$$E(\underline{q}) = \sum_{i=1}^{r=n^2} q_i E_i$$

where E_i are known constant matrices with only one nonzero element, at a different location for different i.

This family of matrices is the so-called "interval matrix" family.

<u>Category M2. Linear dependent variation</u>: For this case, the elements of E vary linearly with the parameters q_i. Thus, we can write

$$E(\underline{q}) = \sum_{i=1}^{r} q_i E_i$$

where E_i are constant, known real matrices. This family is termed the "polytope" of matrices.

<u>Category M3. General variation</u>: For this case

$$E(\underline{q}) = \sum_{i=1}^{r} q_i E_i(\underline{q})$$

Unlike the polynomial case, the matrix case is decidedly more difficult, as evidenced by the lack of strong results. In other words, even though there

are "derivable" necessary and sufficient conditions available [22] for any general type of dependency, these conditions are not really that helpful from the practical application point of view because the conditions are not simple or computationally tractable. It can be safely said that at the present time there are no simple, easily and finitely computable necessary and sufficient conditions available even for the simplest "interval" matrix case. For some special cases of interval matrices, the results of References [23]-[26] are helpful. One natural line of thought for solving this problem (say "interval" matrix stability problem) would be to transform the interval matrix stability problem to the "interval" or "polytope" of polynomial problem and then apply the results discussed previously. However, extreme caution is warranted in this approach in view of the abundance of counter-examples, as discussed in References [27]-[29]. Also, this approach is necessarily conservative (even when it works) as shown by Yedavalli in Reference [30]. In view of these difficulties, much of the literature on the matrix family stability problem is dominated by results which provide "sufficient" conditions for stability.

In the framework of testing the stability of "interval" matrices, Heinen [31] and Argoun [32] present results which use Gershgorin's diagonal dominance concept. Xu [33] imposes certain leading principal minor conditions and Shi and Gao [34] assume "symmetry" (thereby solving a special case). In the framework of obtaining measures of robustness (the results of which can also be applied to test stability), Patel and Toda [16], Yedavalli [35], and Barmish and DeMarco [36] use the Lyapunov method. Presently, Juang et. al. [37] and Qiu and Davison [38] used frequency domain arguments to present sufficient bounds for stability of the interval matrix problem.

For the case of the "polytope" of matrices, Zhou and Khargonekar [39] extend the Lyapunov method used by Yedavalli [19]-[20] to give sufficient conditions for stability and show that the independent variation case of Yedavalli becomes a special case of their result. Thus they get better bounds for the linear dependent variation case. In the present research, in subsection 2.3, we present new results on obtaining sufficient conditions for the linear dependent variation case as well as for the nonlinear case, using the frequency domain results of Reference [37].

2.2 REDUCED CONSERVATISM IN THE TESTING OF HURWITZ INVARIANCE OF UNCERTAIN POLYNOMIALS WITH DEPENDENT COEFFICIENTS

In the classical stability problem of linear time-invariant systems, one is often required to decide whether the zeros of a given polynomial $f(s)$ lie in the strict left half plane. The renowned Routh-Hurwitz criterion [40] has proven to be a valuable tool for this purpose. More recently, however, much attention has been devoted to so-called robustness analysis. Given the motivation that system parameters are often uncertain or changing with time, the fixed polynomial $f(s)$ is replaced by a family of polynomials $\{f(s,q): \ q \in Q\}$ where Q is a compact set and one is faced with the problem of determining whether $f(s,q)$ is stable for all $q \in Q$. In view of the fact that applying the Routh-Hurwitz criterion to each member of the set Q is impractical, a number of alternative methods are proposed as discussed in the previous section. Perhaps most notable are the results of Kharitonov [2], which apply to the special case obtained when the coefficients of $f(s,q)$ vary independently. In this situation, $f(s,q)$ is Hurwitz invariant over Q if and only if four specially constructed "bounding polynomials" are stable.

However, in many control problems, the coefficients of resulting uncertain polynomials $f(s,q)$ usually depend on each other and the Kharitonov test becomes a sufficient condition; as such, it becomes conservative.

In this note we propose a new criterion to reduce the conservatism of the Kharitonov test for Hurwitz invariance of uncertain polynomials. It can be viewed as a generalized version of Kharitonov's results and it is typically suitable for dealing with uncertain polynomials having dependent coefficients. This result is organized as follows: In subsection 2.2.1, we formally define uncertain polynomials and provide some lemmas to be used in later sections. In subsection 2.2.2, we state the main results. Some illustrative examples are given in subsection 2.2.3.

2.2.1 Preliminary Definitions and Lemmas

Definition 2.1: An n-th order uncertain polynomial is defined by

$$f(s,q) = \alpha_0(q)s^n + \alpha_1(q)s^{n-1} + \ldots + \alpha_n(q) \tag{2.1}$$

where q is the uncertainty index vector, Q is a compact subset of R^p, which is the index set for q, α_i: $Q \to R$ are real continuous coefficient functions. $f(s,q)$ is said to be monic if $\alpha_0(q) = 1$. $f(s,q)$ is said to have independent uncertain coefficients if $q = (q_0\ q_1\ \cdots\ q_n)$ and $\alpha_i(q) = q_i$ for all $i = 0,1,\ldots,n$. $f(s,q)$ is said to have dependent uncertain coefficients if there exist $\alpha_i(q) = \phi(\alpha_j(q),q)$ for $i \neq j$.

Definition 2.2: An uncertain polynomial $f(s,q)$ as in (2.1) is said to be Hurwitz invariant over Q if for each $q \in Q$ all the zeros of $f(s,q)$ lie in the strict left half plane.

The following lemma is an immediate consequence of the Routh-Hurwitz theorem. Therefore, no proof is provided.

Lemma 2.1: An uncertain polynomial $f(s,q)$ as in (2.1) with $\alpha_0(q) > 0$ is Hurwitz invariant if and only if the Hurwitz testing matrix

$$H(q) = \begin{bmatrix} \alpha_1(q) & \alpha_3(q) & \alpha_5(q) & 0 & 0 \\ \alpha_0(q) & \alpha_2(q) & \alpha_4(q) & 0 & 0 \\ 0 & \alpha_1(q) & \alpha_3(q) & \alpha_5(q) & 0 \\ 0 & \alpha_0(q) & \alpha_2(q) & \alpha_4(q) & 0 \\ 0 & 0 & \alpha_1(q) & \alpha_3(q) & \alpha_5(q) \end{bmatrix}$$

(illustrated above for $n = 5$) has positive leading principal minors for all $q \in Q_n$.

Lemma 2.1 provides a solid theoretical basis for testing Hurwitz invariance of uncertain polynomials. However, when an uncertain polynomial has high order and its index set Q contains an infinite number of elements, it is impractical to apply the lemma. A computationally feasible test for Hurwitz invariance is needed.

Definition 2.3: Four bounding polynomials of an uncertain polynomial $f(s,q)$ as in (2.1) are defined as follows:

$$f_1(s) = a_0 s^n + a_1 s^{n-1} + b_2 s^{n-2} + b_3 s^{n-3} + a_4 s^{n-4} + \ldots \tag{2.2}$$

$$f_2(s) = b_0 s^n + b_1 s^{n-1} + a_2 s^{n-2} + a_3 s^{n-3} + b_4 s^{n-4} + \ldots \tag{2.3}$$

$$f_3(s) = a_0 s^n + b_1 s^{n-1} + b_2 s^{n-2} + a_3 s^{n-3} + a_4 s^{n-4} + \ldots \tag{2.4}$$

$$f_4(s) = b_0 s^n + a_1 s^{n-1} + a_2 s^{n-2} + b_3 s^{n-3} + b_4 s^{n-4} + \ldots \tag{2.5}$$

where for all $i = 0,1,\ldots,n$,

$a_i = \min\{\alpha_i(q): \ q \in Q\}$, and

$b_i = \max\{\alpha_i(q): \ q \in Q\}$.

Lemma 2.2 (the Kharitonov theorem; see Reference [2] for proof): An uncertain polynomial $f(s,q)$ as in (2.1) is Hurwitz invariant over Q if its four bounding polynomials are Hurwitz. If, in addition, all the coefficients of $f(s,q)$ are independent, then $f(s,q)$ is Hurwitz invariant over Q if and only if its four bounding polynomials are Hurwitz.

Remark: In Reference [2], only monic uncertain polynomials are considered. However, with minor modification, the Kharitonov theorem can be stated in a form as in Lemma 2.2 when $\alpha_0(q)$ is not identity.

Note that in most control problems we are actually dealing with uncertain polynomials having dependent coefficients, and it is often too conservative for Hurwitz invariance to require four Hurwitz bounding polynomials. Our main goal in this section is to provide a methodology to reduce this conservatism.

Main Results

Theorem 2.1 (see Appendix A for proof): An uncertain polynomial $f(s,q)$ as in (2.1) is Hurwitz invariant over Q if and only if there exist positive continuous functions $k_1(q) > 0$, $k_2(q) > 0$ for all $q \in Q$ such that

$$f'(s,q) = \alpha_0'(q)s^n + \alpha_1'(q)s^{n-1} + \ldots + \alpha_n'(q), \quad q \in Q \tag{2.6}$$

where

$$\alpha_i' = \begin{cases} k_1(q)\alpha_i(q), & i \text{ even;} \\ k_2(q)\alpha_i(q), & i \text{ odd.} \end{cases}$$

is Hurwitz invariant over Q.

Theorem 2.1 provides a way to transform an uncertain polynomial to another without changing its Hurwitz invariance. Hence, combining Theorem 2.1 and Lemma 2.2, we immediately have the following theorem.

Theorem 2.2: When positive continuous functions $k_1(q) > 0$, $k_2(q) > 0$ for all $q \in Q$ transform an uncertain polynomial $f(s,q)$ to an uncertain polynomial $f'(s,q)$ as in (2.6), then $f(s,q)$ is Hurwitz invariant over Q if the four bounding polynomials of $f'(s,q)$ are Hurwitz. If, in addition, all the coefficients of $f'(s,q)$ become independent, then $f(s,q)$ is Hurwitz invariant over Q if and only if the four bounding polynomials of $f'(s,q)$ are Hurwitz.

Remark: Theorem 2.2 can be viewed as a generalized version of the Kharitonov theorem. It provides some flexibility in applying the "four bounding polynomial" criterion. By properly choosing $k_1(q)$ and $k_2(q)$, one can reduce the conservatism of the Kharitonov test for uncertain polynomials having dependent coefficients. In the best case, when $k_1(q)$ and $k_2(q)$ are found such that $f'(s,q)$ has independent coefficients, the conservatism is completely reduced. If, however, such $k_1(q)$ and $k_2(q)$ cannot be found, it may still be possible to partially reduce the conservatism by choosing $k_1(q)$ and $k_2(q)$ either to relax the dependence of coefficients or to uniformly reduce the varying bounds of uncertain coefficients. In the latter case, we have the following theorem.

Theorem 2.3 (see Appendix B for proof): If Hurwitz invariance of $f(s,q)$ can be determined by the Kharitonov theorem, it can also be determined by Theorem 2.2, provided $k_1(q)$ and $k_2(q)$ are chosen so that

$$a_i \le a_i', \quad b_i' \le b_i$$

where

$$a_i = \min\{\alpha_i(q): \quad q \in Q\},$$

$$b_i = \max\{\alpha_i(q): \quad q \in Q\},$$

$$a_i' = \begin{cases} \min\{k_1(q)\alpha_i(q): \quad q \in Q\}, \ i \text{ even}; \\ \min\{k_2(q)\alpha_i(q): \quad q \in Q\}, \ i \text{ odd}; \end{cases}$$

and

$$b_i' = \begin{cases} \max\{k_1(q)\alpha_i(q): \quad q \in Q\}, \ i \text{ even}; \\ \max\{k_2(q)\alpha_i(q): \quad q \in Q\}, \ i \text{ odd}. \end{cases}$$

Remark 2.3: Theorem 2.3 indicates a way to choose $k_1(q)$ and $k_2(q)$ which will definitely reduce the conservatism of the Kharitonov test. If the Kharitonov test works, our method also works. If the Kharitonov test fails, our method may still work because the varying bounds are reduced. Generally speaking, one should make his own choice of $k_1(q)$ and $k_2(q)$ for each individual problem. It may also be noted that the reduction in conservatism obtained by the proposed method is in a way similar in concept to the reduction in conservatism obtained by state transformation by Yedavalli and Liang in References [41] and [35] in the context of stability conditions for interval matrices.

Illustrative Examples

Example 2.1: Consider a linear time invariant uncertain system

$$\dot{x}(t) = A(q)x(t) = \begin{bmatrix} 0 & 3 & 1 \\ -1 & q & 0 \\ -1 & -4 & -2 \end{bmatrix} x(t), \quad q \in Q = [-1,1].$$

Determine whether the uncertain system is stable for all $q \in Q$.

This problem is equivalent to testing Hurwitz invariance of the following uncertain polynomial

$$f(s,q) = \det(sI - A(q)) = s^3 + (2 - q)s^2 + (4 - 2q)s + (2 - q).$$

By using the Kharitonov theorem, we test if the following four bounding polynomials are Hurwitz:

$$f_1(s,q) = s^3 + s^2 + 6s + 3$$

$$f_2(s,q) = s^3 + 3s^2 + 2s + 1$$

$$f_3(s,q) = s^3 + 3s^2 + 6s + 1$$

$$f_4(s,q) = s^3 + s^2 + 2s + 3.$$

Obviously, $f_4(s,q)$ is not Hurwitz, hence, the Kharitonov test fails in this problem.

Now, applying Theorem 2.3, we take $k_1(q) = 1$ and $k_2(q) = 1/(2 - q)$ and obtain

$$f'(s,q) = s^3 + s^2 + (4 - 2q)s + 1$$

which has independent coefficients. Since the bounding polynomials of $f'(s,q)$

$$f_1'(s,q) = s^3 + s^2 + 6s + 1$$

and

$$f_2'(s,q) = s^3 + s^2 + 2s + 1,$$

are Hurwitz, we conclude that $f(s,q)$ is Hurwitz invariant and the system is stable for all $q \in Q$.

<u>Example 2.2</u>: Given an uncertain polynomial

$$f(s,q) = s^4 + s^3 + 2qs^2 + s + q, \quad q \in [1.5,4],$$

determine if $f(s,q)$ is Hurwitz invariant.

It is easy to see that one of the bounding polynomials,

$$f(s) = s^4 + s^3 + 3s^2 + s + 4,$$

is not Hurwitz, hence, the Kharitonov test also fails in this example. Now letting $k_1(q) = 1/q$ and $k_2(q) = 1$, we obtain

$$f'(s,q) = (1/q)s^4 + s^3 + 2s^2 + s + 1,$$

which has independent coefficients. Since the bounding polynomials

$$f_1'(s,q) = (2/3)s^4 + s^3 + 2s^2 + s + 1$$

and

$$f_2'(s,q) = (1/4)s^4 + s^3 + 2s^2 + s + 1$$

are Hurwitz, $f(s,q)$ is also Hurwitz invariant.

<u>Example 2.3</u>: Given an uncertain polynomial

$$f(s,q) = s^3 + 2q_1s^2 + (q_1 + q_2)s + q_1q_2, \quad q_1 \in [1,4], \quad q_2 \in [2,4]$$

determine if $f(s,q)$ is Hurwitz invariant.

Since one of the bounding polynomials

$$f(s) = s^3 + 2s^2 + 3s + 16$$

is not Hurwitz, the Kharitonov test fails. We now choose $k_1(q) = 1$ and $k_2(q) = 1/q_1$ and obtain

$$f'(s,q) = s^3 + 2s^2 + (q_1 + q_2)s + q_2, \quad q_1 \in [1,4], \quad q_2 \in [2,4].$$

This polynomial still has dependent coefficients. However, the varying bounds of coefficients are uniformly reduced. The four bounding polynomials of $f'(s,q)$,

$$f_1'(s,q) = s^3 + 2s^2 + 8s + 4,$$

$$f_2'(s,q) = s^3 + 2s^2 + 3s + 2,$$

$$f_3'(s,q) = s^3 + 2s^2 + 8s + 2,$$

$$f_4'(s,q) = s^3 + 2s^2 + 3s + 4$$

are Hurwitz. Hence, $f(s,q)$ is Hurwitz invariant.

2.3 STABILITY ROBUSTNESS MEASURES FOR STATE SPACE MODELS WITH DEPENDENT UNCERTAINTY

In this section, we move on to the case of the time domain state space representation and give improved bounds for stability robustness for systems with dependent uncertainty.

As noted in section 2.1, the aspect of developing explicit upper bounds on the perturbation of linear state space systems to maintain stability has received much attention, in particular, for the problem of "structured uncertainty." In [36], the perturbation matrix elements are taken to vary independently of each other, whereas [39] treats the case in which the uncertain parameters enter linearly into the perturbation matrix. These two techniques use the Lyapunov approach and can also accommodate time-varying (bounded) perturbations. On the other hand, the techniques in [37]-[38] also develop bounds, for independent variations, using a root locus type argument, and obtain a frequency domain based formula for the calculation of the bounds. These techniques are basically aimed at time-invariant perturbations.

In this section we present new bounds for the dependent variation case, in particular for the following two cases: (1) linear dependency, and (2) quadratic variation in a scalar uncertain parameter. The proposed technique includes the bounds of [37]-[38] as a special case. The method is illustrated with the help of several examples.

Consider the state space, perturbed model

$$\dot{x}(t) = (A_0 + E)x(t) \tag{2.7}$$

where A_0 is an $n \times n$ asymptotically stable matrix and E is a "perturbation" matrix.

<u>Independent variations</u> (Category M1): In this case, the elements of the matrix E are assumed to be independent of each other and are bounded such that

$$E_{ij} \leq |E_{ij}|_{max} = \varepsilon_{ij} \quad \text{and} \quad \varepsilon \triangleq \operatorname*{Max}_{i,j} \varepsilon_{ij} \tag{2.8}$$

Denoting Δ as the matrix formed with ε_{ij}

$$\Delta = [\varepsilon_{ij}] \tag{2.9a}$$

we write

$$\Delta = \varepsilon U_e \tag{2.9b}$$

where

$$0 \leq U_{eij} \leq 1 \tag{2.9c}$$

For this case, the bounds on ε_{ij} are obtained in [20] and [36] by a Lyapunov-based approach and in [37]-[38] by a frequency domain based approach. In this section, we are interested in the frequency domain based approach, and hence reproduce the expressions given in [37]-[38] for the above notation. The stability robustness bounds on ε_{ij} are given by [37]-[38] ([38] considers a more general structure also):

$$\varepsilon_{ij} < \frac{1}{\sup\limits_{\omega \geq 0} \rho[\,|(j\omega I - A_0)^{-1}|U_e]} \cdot U_{eij} \quad \text{or} \quad \varepsilon < \mu_{ind} \tag{2.10}$$

where $|(\cdot)|$ denotes the absolute matrix (i.e., matrix with absolute values of the elements) and $\rho(\cdot)$ denotes the spectral radius of the matrix $(\cdot)$ and μ_{ind} denotes the bound for independent variations.

<u>Linear dependent uncertainty</u> (Category M2): In what follows, we consider the case where the uncertain parameters in E are assumed to enter linearly, i.e.,

$$E = \sum_{i=1}^{r} \beta_i E_i \tag{2.11}$$

where E_i are constant, prescribed matrices and β_i are uncertain parameters. Our intention is to give a bound on $|\beta_i|$.

We now present a bound on $|\beta_i|$ and show that the resulting bound specializes to (2.10) for the independent variation case. The proposed bound is less conservative than (2.10) when applied to the situation in which E is given by (2.11) and yields the same bound as in (2.10) when applied to the

independent variation case (when E is given by (2.8)). This is exactly the type of situation that arises in Zhou and Khargonekar [39], where they consider the linear dependency case and specialize it to the independent variation case of Yedavalli, [36].

Remark: It should be mentioned at the outset that it is very important to distinguish between the independent variation case and the dependent variation case at the problem formulation stage. In the independent variation case one gives bounds on ε_{ij} (and consequently on ε), whereas in the dependent variation case, one gives bounds on $|\beta_i|$. This is particularly crucial in the comparison of different techniques. Proper comparison is possible only when the basis, namely whether one is considering the dependent case or the independent case, is established beforehand. It may be noted that the techniques aimed at the independent variation case can accommodate the dependent variation situation, but at the expense of some conservatism; whereas the technique aimed at the dependent case, while it gives a less conservative bound for that case, cannot accommodate the independent variation case (unless it is established, additionally, to handle that situation, as is done in [39] and also later in this section for the new proposed bound).

Theorem 2.4: Consider the system (2.7) with E as in (2.11). Then (2.7) is stable if

$$|\beta_i| < \frac{1}{\sup\limits_{\omega \geq 0} \rho \left(\sum\limits_{i=1}^{r} |(j\omega I - A_0)^{-1} E_i| \right)} = \mu_d \quad \text{for} \quad r > 1 \tag{2.12a}$$

and

$$|\beta_1| < \frac{1}{\sup\limits_{\omega \geq 0} \rho[(j\omega I - A_0)^{-1} E_1]} = \mu_d \quad \text{for} \quad r = 1 \tag{2.12b}$$

where μ_d denotes the bound for "dependent" variation case.

Proof: Given in Appendix C.

It can be shown that the bound (2.10) becomes a special case (2.12a) when one notes that in the independent variation case each E_i will contain a single element and is given by

$$E_{i(n-1)+j} = U_{eij} e_i e_j^T \tag{2.13}$$

where e_i is an n-dimensional column matrix with 1 in the i-th entry and 0 elsewhere. Note that U_{eij} is a scalar and $e_i e_j^T$ is a matrix.

Remark: The bound of (2.12a), when specialized to the independent variation case (i.e., when each E_i contains a single element, at a different place for different i), will be denoted by μ_{ind}. Thus, $\mu_{ind} = \mu_J = \mu_Q$.

Example 2.4: Consider

$$A_0 = \begin{bmatrix} -3 & -2 \\ 1 & 0 \end{bmatrix}$$

and let

$$E = \beta \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \quad \text{(i.e., dependent case)}$$

Yedavalli[20]	μ_Q [37]	μ_J [38]	μ_d (proposed)	Zhou, et.al.[39]
0.236	0.329	0.329	1.0	1.0

If, instead, all the elements in E are assumed to vary independently, then we use

$$E_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad E_2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad E_3 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \quad E_4 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

in the expression (2.12) and get

$$|E_{ij}| < \mu_{ind} = 0.329$$

which is, of course, the same bound as μ_J and μ_Q.

It may be noted that the bound $\mu_d = 1.0$ is also a necessary and sufficient bound if β is varied in the positive direction. However, there was no attempt made in (2.12) to incorporate directional information.

<u>Example 2.5</u>: Consider the same A_0 as in Example 2.4 and let

$$E = \beta \begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix}$$

Yedavalli[20]	μ_Q [38]	μ_J [37]	μ_d (proposed)	Zhou, et.al.[39]
1.0	1.52	1.52	2.0	2.0

<u>Example 2.6</u>: Let us consider the example given in [39] in which the perturbed system matrix is given by

$$(A_0 + BKC) = \begin{bmatrix} -2 + k_1 & 0 & -1 + k_1 \\ 0 & -3 + k_2 & 0 \\ -1 + k_1 & -1 + k_2 & -4 + k_1 \end{bmatrix}$$

Taking the nominally stable matrix to be

$$A_0 = \begin{bmatrix} -2 & 0 & -1 \\ 0 & -3 & 0 \\ -1 & -1 & -4 \end{bmatrix}$$

the error matrix with k_1 and k_2 as the uncertain parameters is given by

$$E = k_1E_1 + k_2E_2$$

where

$$E_1 = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix} \quad \text{and} \quad E_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

The following are the bounds on $|k_1|$ and $|k_2|$ obtained by [25] and the proposed method (corresponding to the bound given by expression (2.9) in [25]):

Yedavalli[20]	μ_Q [38]	μ_J [37]	μ_d (proposed)	Zhou, et.al.[39]
0.815	0.875	0.875	1.75	1.55

It may be noted that, in this result, we do not have counterparts to the bounds given by expressions (2.8) and (2.7) in Zhou and Khargonekar's paper.

<u>Quadratic Variation in a Scalar Parameter</u>: We now consider the case in which a scalar uncertain parameter enters into E in a nonlinear manner, in particular, as a square term; i.e., we assume

$$E_{ij}(q) = k_{ij}q^2 \tag{2.14}$$

where q is the uncertain parameter.

This type of structured uncertainty model occurs in many applications; for example, in [42] where a large-scale interconnected system has the form

$$\dot{x}_1 = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix} x_1 + \begin{bmatrix} 0 & 0 \\ q^2 & 0 \end{bmatrix} x_1 + \begin{bmatrix} 0 & 0 \\ 0 & 2q \end{bmatrix} x_2$$

(2.15a)

$$\dot{x}_2 = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix} x_2 + \begin{bmatrix} 0 & 0 \\ 0 & -2q \end{bmatrix} x_1 + \begin{bmatrix} 0 & 0 \\ q^2 & 0 \end{bmatrix} x_2$$

which can be written as

$$A_0 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & -1 \end{bmatrix} \quad \text{and} \quad E(q) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ q^2 & 0 & 0 & 2q \\ 0 & 0 & 0 & 0 \\ 0 & -2q & q^2 & 0 \end{bmatrix} \tag{2.15b}$$

For a situation of this type, the following iterative method is proposed to obtain an improved bound on $|q|$.

<u>Proposed Iterative Method</u>:

Iteration 1

In this iteration, we ignore the functional dependence and assume the entries in the perturbation matrix vary independently. Accordingly, we let $U_{eij} = 1$ for those entries in which perturbation is present and zero for the other entries. Then compute the bound μ_1 using the expression (2.10). Let the upper bound matrix be denoted by

$$\Delta_{m_1} = \mu_1 U_{ei_1} \tag{2.16}$$

Knowing the elements of Δ_{m_1} and the corresponding functional relationship of the perturbation matrix elements on q in the matrix $[E(q)]_m$ ((•) denotes the modulus matrix), solve for the possible different values of $|q|$ and select the minimum value of $|q|$. Let this value of $|q|$ be denoted by q_{m_1}. With this value of q_{m_1}, compute the matrix $[E(q_{m_1})]_m$ utilizing the functional relationship in the matrix $[E(q)]_m$. Then write

$$[E(q_{m_1})]_m \Delta \varepsilon_2 U_{e2} \tag{2.17}$$

where ε_2 is the maximum modulus element in $[E(q_{m_1})]_m$.

Iteration 2

With ε_2 as the left-hand side of Equation (2.10), compute the bound μ_2 using Equation (2.10) (with U_{e2} replacing the matrix U_e).

If

$$\varepsilon_2 < \mu_2 \tag{2.18}$$

form

$$\Delta_{m_2} = \mu_2 U_{e2} \tag{2.19}$$

and go through the exercise outlined after Equation (2.16) in iteration 1 to obtain q_{m_2}, which will be greater than r_{m_1}. If $\varepsilon_2 \not< \mu_2$, the r_{m_1} becomes the acceptable bound.

<u>Termination</u>: Repeat the iterations until no improvement in the bound q_{m_i} $(i = 1,2,\ldots)$ is observed (say at iteration N) and take q_{m_N} as the acceptable bound.

<u>Example 2.7</u>: Consider the system given in (2.15).

Iteration 1

Let

$$U_{\bullet_1} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{bmatrix}$$

and compute $\mu_1 = 0.4841$. Form

$$\Delta_{m_1} = \mu_1 U_{\bullet_1} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0.4841 & 0 & 0 & 0.4841 \\ 0 & 0 & 0 & 0 \\ 0 & 0.4841 & 0.4841 & 0 \end{bmatrix}$$

Knowing

$$[E(q_m)]_m = \begin{bmatrix} 0 & 0 & 0 & 0 \\ q^2 & 0 & 0 & 2q \\ 0 & 0 & 0 & 0 \\ 0 & 2q & q^2 & 0 \end{bmatrix}$$

we can solve for $|q|$ as i) $|q| = 0.2421$; ii) $|q| = 0.6957$. We take

$$q_{m_1} = \mathrm{Min}[0.2421, 0.6957] = 0.2421.$$

We then form the matrix $[E(q_{m_1})]_m$, i.e.,

$$[E(q_{m_1})]_m = \begin{bmatrix} 0 & 0 & 0 & 0 \\ (0.2421)^2 & 0 & & 2(0.2421) \\ 0 & 0 & 0 & 0 \\ 0 & 2(0.2421) & (0.2421)^2 & 0 \end{bmatrix}$$

$$= 0.4841 \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0.121 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0.121 & 0 \end{bmatrix}$$

$$= \epsilon_2 U_{e_2} \qquad \text{(Thus } \epsilon_2 = 0.4841.\text{)}$$

<u>Iteration 2</u>: Compute the bound μ_2 with U_{e_2} as the "structured" matrix and obtain

$$\mu_2 = 0.8921.$$

Noting that $\mu_2 > \epsilon_2$ (and that $\mu_2 > \mu_1$), we proceed further and form

$$\Delta_{m_2} = \mu_2 U_{e_2} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0.1079 & 0 & 0 & 0.8921 \\ 0 & 0 & 0 & 0 \\ 0 & 0.8921 & 0.1079 & 0 \end{bmatrix}$$

Comparing Δ_{m_2} with $[E(q)]_m$, we can solve for $|q|$ as i) $|q| = 0.3285$ (from q^2 term), ii) $|q| = 0.446$ (from $2q$ term). We take

$$q_{m_2} = \text{Min}[0.3285, 0.446] = 0.3285.$$

Thus one acceptable range of q is

$$0 \leq |q| \leq 0.3285.$$

One can carry out these iterations further to obtain an improved bound on $|q|$.

III. SYNTHESIS OF ROBUST CONTROLLERS FOR LINEAR SYSTEMS WITH STRUCTURED UNCERTAINTY

In the previous section, the results obtained were essentially "analysis" tools. No attempt was made to synthesize controllers. Of course, an exercise in that type of analysis is helpful before one can design controllers for a given objective. Aided with this type of analysis, in this section we emphasize the aspect of design and present two design methods, one in the state space framework and the other in the transfer function framework. In the next section, the system is described by a state space representation and the aim is to design a controller that maximizes the stability robustness bounds for a given structured uncertainty with adequate control effort considerations. The design is applicable to systems with time varying perturbations. In section 3.2, the uncertain system is assumed to be represented in transfer function form, parametrized by an uncertain parameter vector. A single compensator that simultaneously stabilizes the system is sought. The conditions under which this compensator exists are stated along with the construction of the compensator. The design is shown to be suitable for a class of non-minimum phase systems also.

3.1 TIME DOMAIN CONTROL DESIGN FOR ROBUST STABILIZATION AND REGULATION

There is a considerable amount of literature on the aspect of designing <u>linear</u> controllers for linear time invariant systems with small parameter uncertainty. However, for uncertain systems whose dynamics are described by

interval matrices (i.e., matrices whose elements are known to vary within a given bounded interval), linear control design schemes that guarantee stability have been relatively scarce. Vinkler and Wood [43] compare several techniques for designing linear controllers for robust stability for a class of uncertain linear systems. Among the methods considered are the standard linear quadratic regulator (LQR) design, the guaranteed cost control (GCC) method of Chang and Peng [15], and the multistep guaranteed cost control (MGCC) of Vinkler and Wood. In these methods, the weighting on state in a quadratic cost function and the Riccati equation are modified in the search for an appropriate controller. Also the parameter uncertainty is assumed to enter linearly and restrictive conditions are imposed on the bounding sets. In Chang and Peng, norm inequalities on the bounding sets are given for stability but they are conservative since they do not take advantage of the system structure. There is no guarantee that a linear state feedback controller exists. Thorp and Barmish [44] utilize the concept of "matching conditions" (MC), which in essence constrain the manner in which the uncertainty is permitted to enter into the dynamics and show that a linear state feedback control that guarantees stability exists provided the uncertainty satisfies matching conditions. By this method large bounding sets produce large feedback gains but the existence of a linear controller is guaranteed. But no such guarantee can be given for general "mismatched" uncertain systems. Hollot and Barmish [45] and Schmitendorf [46] present methods which need the testing of definiteness of a Lyapunov matrix obtained as a function of the uncertain parameters. Ackermann [47], in the multimodel theory approach, considers a discrete set of points in the parameter uncertainty range to establish the stability. This section addresses the

stabilization problem for a continuous range of parameters in the uncertain parameter set (i.e., in the context of interval matrices). The proposed approach attacks the stability of the interval matrix problem directly in the matrix domain rather than converting the interval matrix to interval polynomials and then testing the Kharitinov polynomials (Soh and Evans [48]). Towards this direction, bounds on the individual elements of the perturbation matrix are developed in Yedavalli [20] which are shown to be less conservative than the existing measures since the method utilizes the structural information of the uncertainty, and are applicable to systems with time varying perturbations. In Yedavalli [49], a control design method is presented in which the linear state feedback gain is obtained by the standard Riccati equation and then the robustness of the gain is investigated by the resulting elemental perturbation bound. In this section we design the gain such that its determination directly uses the uncertainty structure; this is done by parameter optimization. In other words, the controller gains are determined such that they maximize (in a certain sense) the elemental perturbation bound for a given uncertainty structure. The design technique is then extended to the observer based feedback case as considered in Menga and Dorato [50] and the relative tradeoffs are discussed.

Elemental Perturbation Bounds for Robust Stability

In this section, we first briefly review the upper bounds for robust stability presented in Reference [20] for "structured" (elemental) perturbation. Structured perturbations are those for which magnitude bounds on the individual matrix elements are known for a given model structure.

Consider the following linear dynamic system

$$\dot{x}(t) = A(t)x(t) \tag{3.1a}$$

$$= [A_0 + E(t)]\, x(t) \tag{3.1b}$$

where $x(t) \to R^n$ is the state vector. A_o is the $n \times n$ nominally stable matrix and $E(t)$ is the "error" matrix. In the case of structured perturbation, the elements of $E(t)$ are such that

$$\underset{\forall t}{\text{Max}} |E_{ij}(t)| = \varepsilon_{ij} \quad \text{and} \quad \varepsilon = \underset{\forall i,j}{\text{Max}}\ \varepsilon_{ij} \tag{3.2}$$

In Reference [20], it is shown that the system (3.1) (with (3.2)) is asymptotically stable if

$$\varepsilon_{ij} < \frac{1}{\sigma_{max}(P_m U_e)_s} \cdot U_{eij} = \mu U_{eij} \tag{3.3a}$$

for all $U_{eij} \neq 0$, $i,j = 1,\ldots,n$, where P satisfies the Lyapunov matrix equation

$$PA_o + A_o^T P + 2I_n = 0 \tag{3.3b}$$

and

$$U_{eij} \triangleq \varepsilon_{ij}/\varepsilon \quad (\text{Thus } 0 < U_{eij} < 1) \tag{3.3c}$$

Here $\sigma(\cdot)$ denotes the singular value of $(\cdot)$, $(\cdot)_s$ denotes the symmetric part of the matrix $(\cdot)$, and $(\cdot)_m$ denotes the matrix formed with the absolute values of the entries of the matrix $(\cdot)$. Simple examples illustrating this bound and the role of matrix U_e in utilizing the structural information about the error matrix are given in Reference [20].

It may be noted that U_e can be formed even if one knows only the ratio $\varepsilon_{ij}/\varepsilon$ instead of knowing the two terms separately. One suitable choice for the ratio is

$$U_{eij} = \varepsilon_{ij}/\varepsilon = |A_{oij}|/|A_{oij}|\max \tag{3.3d}$$

for all i,j for which $\varepsilon_{ij} \neq 0$.

Remark: From (3.2), it is seen that ϵ_{ij} are the maximum modulus deviations expected in the individual elements of the nominal matrix A_o. If we denote the matrix Δ as the matrix formed with ϵ_{ij}, then clearly Δ is the "majorant" matrix of the actual error matrix E(t). It may be noted that U_e is simply the matrix formed by normalizing the elements of Δ (i.e., ϵ_{ij}) with respect to the maximum of ϵ_{ij} (i.e., ϵ).

$$\text{i.e., } \Delta = U_e \quad \text{(absolute variation)} \tag{3.4}$$

Thus ϵ_{ij} here are the absolute variations in A_{oij}. Alternatively one can express Δ in terms of percentage variations with respect to the entries of A_{oij}. Then one can write

$$\Delta = \delta A_{om} \quad \text{(relative or percentage variation)} \tag{3.5}$$

where $A_{omij} = |A_{oij}|$ for all those i,j in which variation is expected and δ_{ij} are the maximum relative variations with respect to the nominal value of $|A_{oij}|_{max}$ and

$$\delta = \max_{i,j} \delta_{ij}.$$

Clearly, one can then get a bound on δ for robust stability as

$$\delta < \frac{1}{\sigma_{max}[P_m A_{om}]_s} = \mu_r \tag{3.6}$$

where P is the same as in (3.3b).

We now define, as a measure of stability robustness, an index called stability robustness index β_{SR} as follows:

Case A: LHS of (3.3) or (3.6) is known (i.e., checking stability for given perturbation range). For this case,

$$\beta_{SR} \Delta \mu - \epsilon \quad (\text{or } \mu_r - \delta). \tag{3.7}$$

Thus $\beta_{SR} > 0$ corresponds to the stability robustness region.

Case B: LHS of (3.3) or (3.6) is not known (i.e., specifying the bound). For this case,

$$\beta_{SR} \Delta \mu \quad (\text{or } \mu_r). \tag{3.8}$$

Using the concept of state transformation, it has been shown in Yedavalli and Liang [41] that it is possible to further improve the bounds on ε. For structured perturbation, it is possible to get higher bounds even with the use of a diagonal transformation. This result can be stated as follows.

Given the diagonal transformation matrix

$$\bar{M} = \text{Diag} \quad [m_1, m_2, \ldots, m_n]$$

the system of (3.1) is stable if

$$\varepsilon_{ij} < \frac{\hat{\mu}_s}{\max\limits_{i,j} \left| \frac{m_j}{m_i} U_{eij} \right|} \qquad U_{eij} = \hat{\mu}_s U_{eij}$$

or

$$\varepsilon < \hat{\mu}_s$$

where

$$\hat{\mu}_s = 1/\sigma_{max}(\hat{P}_m \hat{U}_e)_s$$

and

$$\hat{P}\hat{A}_o + \hat{A}_o^T \hat{P} + 2I_n = 0$$

and U_e is formed such that

$$\hat{U}_{eij} = \frac{\hat{\varepsilon}_{ij}}{\hat{\varepsilon}} \quad \text{where} \quad \hat{\varepsilon}_{ij} = \left| \frac{m_j}{m_i} \right| \mu_{eij}$$

and

$$\hat{\varepsilon} = \hat{\varepsilon}_{ij_{max}}, \quad \hat{A}_o = \bar{M}^{-1} A_o \bar{M}.$$

(Note that in general $\hat{U}_e \neq \bar{M}^{-1} U_e \bar{M}$.)

Extension to Closed Loop Systems

Consider the linear, time invariant system described by

$$\dot{x} = Ax + Bu \qquad x(0) = x_0 \tag{3.9}$$

where x is an $n \times 1$ state vector, and the control u is $m \times 1$. The matrix pair (A,B) is assumed to be completely controllable.

For this case, the nominal closed loop system matrix is given by

$$\bar{A} = A + BG, \quad G = -R_0^{-1}B^TK/\rho_c \tag{3.10a}$$

and

$$KA + A^TK - KB\frac{R_0^{-1}}{\rho_c}B^TK + Q = 0, \tag{3.10b}$$

and $\bar{A}$ is asymptotically stable.

Here G is the Riccati based control gain where Q, R_o are any given weighting matrices which are symmetric and positive definite and ρ_c is the design variable.

The main interest in determining G is to keep the nominal closed loop system stable. The reason the Riccati approach is used to determine G is that it readily renders (A + BG) asymptotically stable with the above assumption on Q and R_o.

Now consider the perturbed system with linear time varying perturbations $E_A(t)$ and $E_B(t)$, respectively, in the matrices A and B, i.e.,

$$\dot{x} = [A + E_A(t)]x(t) + [B + E_B(t)]u(t) \tag{3.11}$$

Let ΔA and ΔB be the perturbation matrices formed by the maximum modulus deviations expected in the individual elements of matrices A and B, respectively. Then one can write

$$\left.\begin{aligned} \Delta A &= \varepsilon_a U e_a \\ \Delta B &= \varepsilon_b U e_b \end{aligned}\right\} \quad \text{(Absolute variation)} \tag{3.12}$$

where ε_a is the maximum of all the elements in ΔA and ε_b is the maximum of all elements in ΔB. Then the total perturbation in the linear closed loop system matrix of (3.10) with nominal control $u = Gx$ is given by

$$\Delta = \Delta A + \Delta B G_m = \varepsilon_a U e_a + \varepsilon_b U e_b G_m \tag{3.13}$$

Assuming the ratio $\varepsilon_b/\varepsilon_a = \bar{\varepsilon}$ is known, we can extend the main result of (3.3) to the linear state feedback control system of (3.9) and (3.10) and obtain the following design observation.

<u>Design Observation 1</u>: The perturbed linear system is stable for all perturbations bounded by ε_a and ε_b if

$$\varepsilon_a < \frac{1}{\sigma_{max}[P_m(Ue_a + \varepsilon U e_b G_m)]_s} = \mu \tag{3.14a}$$

and $\varepsilon_b < \bar{\varepsilon}\mu$ where

$$P(A + BG) + (A + BG)^T P + 2I_n = 0 \tag{3.14b}$$

Alternatively, we can write

$$\Delta A = \delta_a A_m$$

$$\text{(Absolute variation)} \tag{3.15}$$

$$\Delta B = \delta_b B_m$$

where $Am_{ij} = |A_{ij}|$ and $Bm_{ij} = |B_{ij}|$ for all i,j in which variation is expected and $Am_{ij} = 0$, $Bm_{ij} = 0$ for all i,j in which there is no variation expected. For this situation, assuming $\delta_b/\delta_a = \bar{\delta}$ is known, we get the following bound on δ_a for robust stability.

<u>Design Observation 2</u>: The perturbed linear system is stable for all relative (or percentage) perturbations bounded by δ_a and δ_b if

$$\delta_a < \frac{1}{\sigma_{max}[P_m(A_m + \bar{\delta} B_m G_m)]_s} = \mu_r \tag{3.16}$$

and $\delta_b < \bar{\delta}\mu_r$ where P satisfies (3.14b).

Note that the above expressions can be suitably modified if only either ΔA or ΔB is present.

Remark: If we suppose $\Delta A = 0$, $\Delta B = 0$ and expect some control gain perturbations ΔG, where we can write

$$\Delta G = \varepsilon_g U e_g \tag{3.17a}$$

then stability is assured if

$$\varepsilon_g < \frac{1}{\sigma_{max}(P_m B_m U e_g)_s} = \mu g \tag{3.17b}$$

In this context, μ_g can be regarded as a "gain margin."

For a given ε_{aij} and ε_{bij}, one method of designing the linear controller would be to determine G of (3.10) by varying ρ_c and pick a control gain that satisfies (3.14). If ε_a and ε_b are not known a priori, then one can determine ρ_c of (3.10) such that μ is maximum. For an aircraft control example which utilizes this method, see Reference [49].

Control Design with Stability Robustness Constraint

In the previous section, efforts were directed to design a linear full state feedback controller for robust stability. However, in that treatment, the control gain determination does not directly involve the stability robustness criterion as a design constraint. Instead, for a predetermined linear control gain (obtained by many different nominal methods), the perturbation bound is calculated and in the cases where the parameter perturbation ranges are given, the stability robustness condition is checked (for robust stability). Even though the bound μ utilizes the structural information of the uncertainty, this design procedure does not utilize the structural information U_e in the determination of the control gain G.

In this section, we attempt to solve the problem of control design for robust stabilization in a more direct and general way by formulating it as a parameter optimization problem. Instead of designing the control gains by nominal means and then checking its stability robustness bounds, we propose to include the stability robustness condition explicitly in the design procedure as a performance measure. In this way it is possible to exploit (in principle) the uncertainty structure U_e in the design procedure.

Performance Index Specification

An optimization problem to maximize the stability robustness bound μ can be posed as follows. (For simplicity, let us consider the case $\Delta B = 0$.)

Minimize

$$J_1 = \sigma_{max}(P_m U_{ea})_s$$

i.e., maximize (3.18a)

$$\mu = \frac{1}{\sigma_{max}(P_m U_{ea})_s}$$

subject to the constraints

$$P(A + BG) + (A + BG)^T P + 2I_n = 0 \tag{3.18b}$$

and

$$\mathrm{Re}\lambda_i(\bar{A}) = \mathrm{Re}\lambda_i(A + BG) < 0 \tag{3.18c}$$

where $\lambda(\cdot)$ are the eigenvalues of $(\cdot)$ and

$$u = Gx \tag{3.18d}$$

We now append the above stability robustness measure to the standard quadratic performance index in state x and control u to take the control effort constraints into consideration. Thus the new performance index would be

$$\bar{J}_1 = \sigma_{max}(P_m U_{ea})_s + \left[\int_0^\infty (x^T Q x + u^T R u)\, dt \right]$$

$$= \sigma_{max}(P_m U_{ea})_s + 1/2 \text{ trace } KX_0; \quad R = \rho_c R_0, \quad \text{and} \quad Q > 0 \tag{3.19}$$

where K satisfies the Lyapunov equation

$$K(A + BG) + (A + BG)^T K + G^T R G + Q = 0 \tag{3.20a}$$

and

$$X_0 = x_0 x_0^T \tag{3.20b}$$

The optimization problem then is as follows: Find G such that the performance index

$$\bar{J}_1 = [\sigma_{max}(P_m U_{ea})_s + 1/2 \text{ trace } KX_0] \tag{3.21a}$$

is minimized subject to the constraints

$$P(A + BG) + (A + BG)^T P + 2I_n = 0 \tag{3.21b}$$

$$K(A + BG) + (A + BG)^T K + G^T R G + Q = 0 \tag{3.21c}$$

and

$$\mathrm{Re}\{\lambda_i(A + BG)\} < 0 \tag{3.21d}$$

<u>Modified Performance Index</u>

Note that the above performance index $\bar{J}_1$ contains a term involving the maximum singular value as well as a positive matrix P_m. Even though there are algorithms to optimize $\sigma_{max}(\cdot)$, optimization of an index like the one posed $(P_m U_e)$ is a formidable task as it is computationally very complex. Hence we intend to modify the performance index such that it becomes more tractable.

Noting that the Frobenius norm of a matrix is always an upper bound on the spectral norm of the matrix, i.e.,

$$||(\cdot)||_F > \sigma_{max}(\cdot)_s \tag{3.22}$$

and that

$$\sigma_{max}(\cdot) > \sigma_{max}(\cdot)_s \tag{3.23}$$

we propose the following upper bound J_s to be minimized instead of $\sigma_{max}(P_m U_{es})_s$.

<u>Proposition 3.1</u>:

$$J_s = 1/2 \text{ trace } (PWP^T + P^T WP) > \sigma^2_{max}(P_m U_e)_s \tag{3.24}$$

for some suitable diagonal weighting matrix W.

The diagonal weighting matrix W is such that $W_{ii} = 0$ whenever $U_{eij}(j = 1,2,\ldots,n) = 0$ for a given row i and $W_{ii} = w_i$ whenever U_{eij} $(j = 1,2,\ldots,n) \neq 0$ for a given row i and any column j. Even though the specification of w_i is crucial in establishing the upper bound property of J_s as in (3.24), it turns out that it is possible to specify the $w_i > 0$ as arbitrary and transfer its implication in the design to another design variable, namely, ρ_c, the weighting on the control variable.

<u>Remark</u>: At this stage of research, one limitation of specifying the W matrix as above is that it reflects the uncertainty structure (U_e) only partially in the sense that w_i is the same (i.e., the same diagonal entry) irrespective of whether there are uncertain elements present in different j-th locations or only in one j-th location. However, for those uncertainty structures U_e which make $U_e U_e^T$ diagonal, we can replace

$$W = \alpha U_e U_e^T \quad \text{(diagonal)}(\alpha \text{ is a scalar } > 0) \tag{3.25}$$

in (3.24), which then amounts to utilizing the structure of the uncertainty completely and α acts as a weighting parameter. The forms of U_e which render $U_e U_e^T$ diagonal are given by:

<u>Case A</u>: variations in one row only

<u>Case B</u>: variations in diagonal elements

<u>Case C</u>: variations in antidiagonal elements

Case D: no two varying elements are in the same column. In fact, Cases A, B, and C are special cases of Case D.

Efforts are underway to prescribe a more versatile performance index such that it completely utilizes the uncertainty structure and still maintain the tractability property.

We are now in a position to state the problem of finding the "optimal" state feedback gain G for robust stability and nominal regulation as follows: Minimize

$$J = \{1/2 \text{ trace } [PWP^T + P^TWP] + 1/2 \text{ trace } KX_0\} \tag{3.26a}$$

subject to

$$P(A + BG) + (A + BG)^TP + 2I_n = 0 \tag{3.26b}$$

$$K(A + BG) + (A + BG)^TK + G_\rho^TR_0G + Q = 0 \tag{3.26c}$$

$$R_e\lambda_i(A + BG) < 0 \tag{3.26d}$$

Solution by Parameter Optimization

We approach the solution to the above nonlinear (quadratic performance index) programming problem by writing down necessary conditions and investigating the solutions which satisfy them. Using the technique of Lagrange multipliers, we transform the above constrained optimization problem to an unconstrained optimization problem by defining the Hamiltonian. Thus we write minimize

$$\{H\} \tag{3.27}$$

where H is the Hamiltonian given by

$$H = \text{trace } \{1/2\ (P^TWP + PWP^T) + 1/2\ KX_0 + L_1(P\bar{A} + \bar{A}^TP + 2I_n) + L_2(K\bar{A} + \bar{A}^TK + G^TRG + Q)\} \tag{3.28}$$

and L_1 and L_2 are the Lagrange multiplier matrices corresponding to the two matrix constraints.

The first order necessary conditions are given by

$$\frac{\partial H}{\partial L_1} = (A + BG)^T P + P(A + BG) + 2I_n = 0 \tag{3.29a}$$

$$\frac{\partial H}{\partial L_2} = (A + BG)^T K + K(A + BG) + G^T RG + Q = 0 \tag{3.29b}$$

$$\frac{\partial H}{\partial P} = (A + BG)L_1^T + L_1^T(A + BG)^T + PW + WP = 0 \tag{3.29c}$$

$$\frac{\partial H}{\partial K} = (A + BG)L_2^T + L_2^T(A + BG)^T + X_0^T = 0 \tag{3.29d}$$

$$\frac{\partial H}{\partial G} = 2B^T(PL_1 + KL_2) + 2RGL_2 = 0 \tag{3.29e}$$

In arriving at these conditions, the matrix derivative identities given in Athans [50] are used.

One can determine the gain G by simultaneously solving for the above equations, starting with an initial guess G_o. Guidelines for obtaining solutions to the above type of equations are given in Levine and Athans [51].

<u>Remark</u>: Note that when the stability robustness constraint is absent (which is the case by making $W = 0$), the above problem formulation reduces to the standard linear quadratic regulator problem and the equations (3.29) yield the standard Riccati equation for the optimal control gain G.

However, with the stability robustness constraint present, the gain G is seen to be a function of the initial condition matrix X_o. As pointed out in Levine and Athans [52], this dependence of the controller on the initial condition can be removed by treating x_o to be a random vector with zero mean and uniformly distributed over a sphere of unit radius, thereby considering the worst case situation. Accordingly, we can modify the performance index as

$$J = (1/2 \text{ trace } [PWP^T + P^TWP] + 1/2 \text{ trace } K) \tag{3.30}$$

where it is assumed that $X_o = \xi(x_o x_o^T) = I_n$ (with ξ being the expectation operator).

Accordingly, one of the necessary conditions, (3.29d), changes to

$$(A + BG)L_2^T + L_2^T(A + BG)^T + I_n = 0 \tag{3.31}$$

which then allows us to express the "optimal" control given G explicitly, using (3.29e) as

$$G = -R^{-1}B^T(PL_1 + KL_2)L_2^{-1} \tag{3.32}$$

<u>Extension to Observer Based Feedback Controller</u>

It may be noted that the above procedure can be readily extended to the case of observer based feedback control, with the system description given by

$$\dot{x} = [A + E_A(t)]x(t) + [B + E_B(t)]u(t) \tag{3.33a}$$

$$z = [M + E_m(t)]x(t) \tag{3.33b}$$

where z is the ℓ-vector of measurements and E_A, E_B, and E_M are the "perturbations" in the nominal matrices A, B, and M.

In this connection a few remarks about the paper by Menga and Dorato [52] are in order. The problem formulation of designing an observer based controller in this section is similar in spirit to that of Menga and Dorato. However, in that paper no explicit bounds on the individual elements of the perturbation are incorporated as is done here. Also, some restrictions are placed on the uncertainty structure to fit it into their proposed problem formulation (such as orthogonality of the uncertainty matrices, and only a single uncertain parameter being allowed in E_B). The major contribution of the present work that is significantly different from that of Menga and Dorato is the exploitation of the uncertainty structure in obtaining the stability robustness bounds. This in turn results in the consideration of two separate

Lyapunov equations in the problem formulation (Equation (3.26)) as opposed to only one Lyapunov equation considered in Menga and Dorato.

The observer structure is the standard Luenberger observer (Reference [53]), with

$$\dot{\beta} = F\beta + Hu + Dz \tag{3.34}$$

where β is the estimate of x and the matrices F, D, and H satisfy the observer conditions

$$SA - DM = FS \tag{3.35a}$$

$$H = SB \tag{3.35b}$$

for an appropriate transformation matrix S. The control is given by

$$u = G\beta \tag{3.36}$$

For brevity, the details of the problem formulation, which follows the development given in previous sections, are not given here.

<u>Example 3.3</u>: Consider a simple second order linear time invariant system given by

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ a & -0.5 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u \tag{3.37}$$

where a is the uncertain parameter with nominal value $\bar{a} = 1$. Notice that

$$U_{aa} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}.$$

We select

$$W = \alpha U_{\bullet} U_{\bullet}^{T} = \begin{bmatrix} 0 & 0 \\ 0 & \alpha \end{bmatrix} \quad (\alpha \geq 1)$$

It may be noted that the robustness weighting matrix W incorporates the uncertainty structure (that only a_{21} element is varying) in an explicit way.

Now with $Q = I_2$, $R = \rho R_o = \rho$ (i.e., $R_o = 1$ and ρ as a design variable) and $\alpha = 1$ for "robust" design and $\alpha = 0$ for nominal design, we can get the "optimal" control gain G by following the proposed procedure.

The comparison of "robust (PO) state feedback control law" versus the "nominal state feedback control law" is depicted in Figure 1, where the perturbation bound μ_{21} is plotted against the nominal control effort

$$J_{un} = \left(\int_0^{\infty} u^T u dt \right)^{1/2}.$$

As anticipated for a given control effort, the robust control law yields a higher perturbation bound μ_{21} than the nominal control law, indicating the usefulness of the proposed optimization procedure. Also, since in this case the "matching condition" is satisfied, it is seen that the higher the control effort, the greater is the perturbation bound.

In the next section, we present a design procedure for systems represented by transfer functions with uncertain coefficients.

3.2 FREQUENCY DOMAIN CONTROL DESIGN FOR ROBUST STABILIZATION (INCLUDING A CLASS OF NON-MINIMUM PHASE SYSTEMS)

Much of the motivation for the transfer function based frequency domain design work comes from the so-called simultaneous stabilization problem introduced in References [54] and [55]. Given plants $P_0(s)$, $P_1(s), \ldots, P_k(s)$,

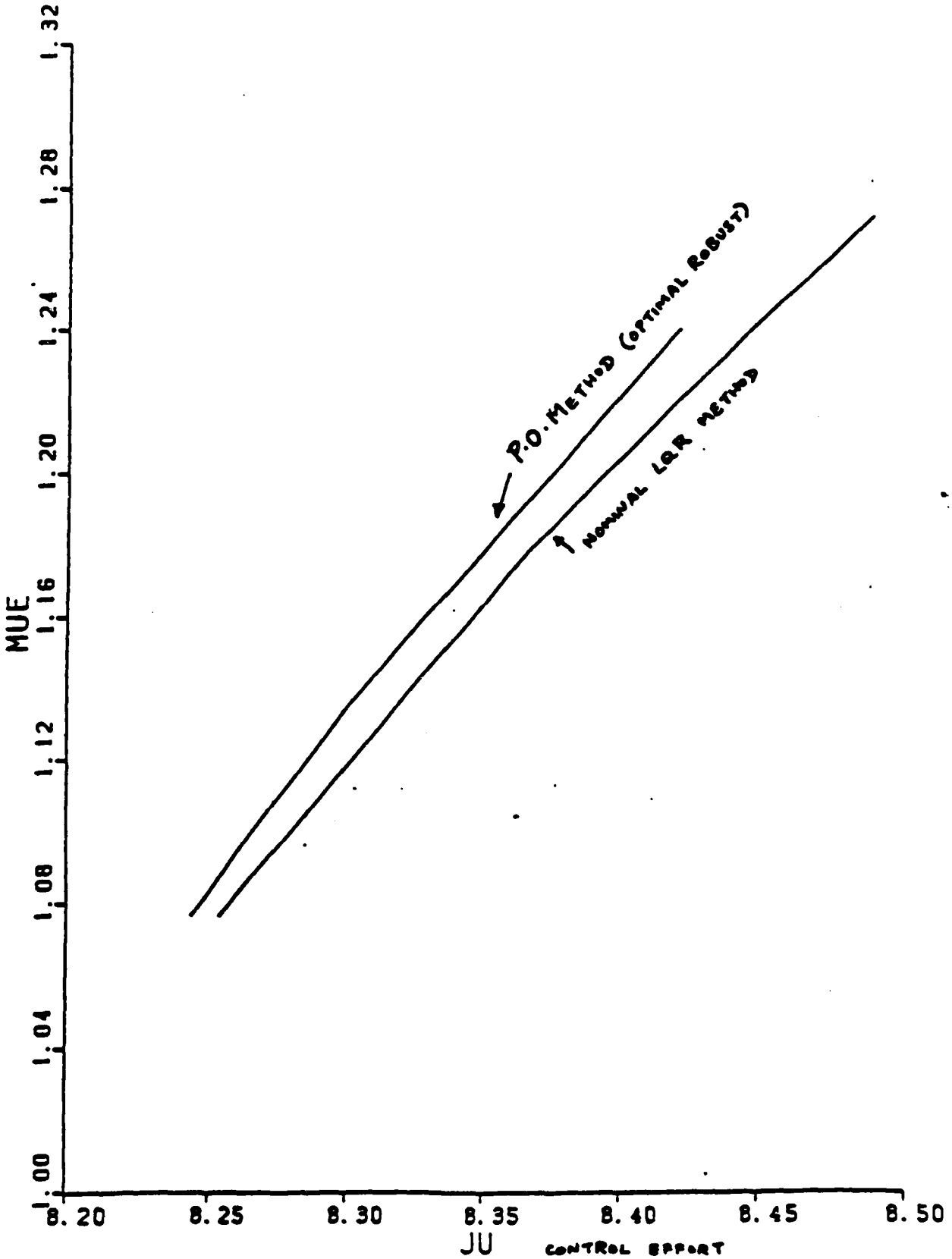
MUE
1.00
1.04
1.08
1.12
1.16
1.20
1.24
1.28
1.32
8.20
8.25
8.30
8.35
8.40
8.45
8.50
JU
CONTROL EFFORT
P.O. METHOD (OPTIMAL ROBUST)
NOMINAL LQR METHOD

FIG 1

does there exist a single compensator C(s) that stabilizes all of them? In their paper, Saeks and Murray [54] develop geometric conditions for simultaneous stabilization and state that their solution is "mathematical in nature and not intended for computational implementation." They indicate that computational criteria are only known for the two plant case (Reference [56]). The subsequent work of Vidyasagar and Viswanadham [55] is concerned with a multi-input multi-output (MIMO) generalization of some of the single-input single-output (SISO) results of Reference [54]. To this end, they prove that the problem of simultaneously stabilizing k+1 plants is equivalent to the problem of simultaneously stabilizing k plants with the added requirement that the compensator itself must be stable. As far as computational criteria are concerned, the results in Reference [55] imply a complete solution for the two plant case; i.e., upon reducing the two plant problem to that of finding a stable compensator for a single plant, one can apply the results of Youla, Bongiorno, and Lu [57]. To illustrate, if $P_0(s)$ and $P_1(s)$ are strictly proper SISO transfer functions with $P_0(s)$ stable, then the results of Vidyasagar and Viswanadham lead to the requirement that the "difference plant," $P_1(s) - P_0(s)$, be stabilizable via a stable compensator. Hence, according to Reference [57], the problem reduces to checking for satisfaction of the parity interlacing property. Namely, we examine the pole zero pattern of $P_1(s) - P_0(s)$ and require that no zeros on the non-negative real axis lie to the left of an odd number of real poles, multiple poles counted according to their multiplicity.

It is also shown in Reference [55] that given two $n \times m$ plants, one can generically stabilize them simultaneously, provided that either n or m is greater than one. This result is further generalized in Reference [58], where

it is shown that general simultaneous stabilizability of r $n \times m$ plants is guaranteed if $\max(n,m) \geq r$.

In view of the results of Reference [55], the issue of finding a computationally feasible test for simultaneous stabilizability (for three or more plants) was raised again in a paper by Emre [59]. In his work, SISO plants are considered, and the problem of finding a computational test is solved for the special case obtained by imposing a constraint that all k+1 closed loop systems must end up having the <u>same</u> characteristic polynomial.

In Reference [60], Barmish and Wei derive sufficient conditions under which a family of SISO systems can be simultaneously stabilized by a proper (or strictly proper if desired) stable compensator. Regularity conditions are imposed on the plant family coefficients, and it is assumed that the plant family is <u>minimum phase</u> with one sign high frequency gain. A computation procedure for constructing a robust compensator is also provided. These results have been further generalized for MIMO systems by Wei and Barmish [61].

In this section, we intend to relax the minimum phase requirement and show that there exist certain classes of non-minimum phase systems that can be simultaneously stabilized by a single compensator.

<u>Preliminary Definitions and Lemmas</u>:

<u>Definition 3.1</u>

A family of polynomials with different degrees is represented by an *uncertain polynomial* of the form

$$f(s,q) = \sum_{i=0}^{d(q)} \alpha_i(q) s^{d(q)-i}, \quad q \in Q \tag{3.38}$$

where Q is a subset of R^p, which is an *index set*, $\alpha_i(\cdot)$: $Q \to R$ are *coefficient functions* and $d(\cdot)$: $Q \to \{0,1,2,\ldots\}$ is the *degree function*.

An uncertain polynomial f(s,q), as in (3.38), is said to be *Hurwitz invariant* (over Q) if for each $q \in Q$, all the zeros of f(s,q) lie in the strict left half plane. f(s,q) is said to be *positive Hurwitz invariant* (PHI), over Q, if f(s,q) is Hurwitz invariant and $\alpha_0(q) > 0$ for all $q \in Q$. Finally, f(s,q) is said to be *negative Hurwitz invariant* (NHI), over Q, if -f(s,q) is positive Hurwitz invariant (over Q).

<u>Definition 3.2</u>

An uncertain polynomial f(s,q), as in (3.38) is said to be *standard* if it satisfies the following conditions:

(1) Compact index set: The index set Q is a compact subset of R^p.

(2) Bounded degree: $d(\cdot)$ is bounded over Q, i.e.,

$$d_{max} = \max\{d(q): \ q \in Q\} < +\infty.$$

(3) Continuous coefficients: Given any possible degree $n \leq d_{max}$, it follows that $\alpha_0(\cdot),\alpha_1(\cdot),\ldots,\alpha_n(\cdot)$ are continuous over Q.

(4) Closed degree sets: Given any possible degree $n \leq d_{max}$, it follows that the degree set

$$Q_n = \{q \in Q: \ d(q) = n\}$$

is a closed subset of Q. In view of the fact that Q is assumed to be compact, closedness of Q_n implies that Q_n is compact; i.e., a closed subset of a compact set is compact.

Remark: The condition of closed degree sets has the effect of partitioning a family of polynomials into closed subsets each having its own distinct degree. A simple example which violates the condition is

$$f(s,q) = qs^2 + s + 1; \ q \in [0,1].$$

With $Q = [0,1]$, we obtain $Q_1 = [0]$, but $Q_2 = (0,1]$, which is not closed.

Definition 3.3

An *uncertain proper rational function* is defined to be a ratio of the form

$$P(s,q) = f_1(s,q)/f_2(s,q), \quad q \in Q \tag{3.39}$$

where $f_1(s,q)$ and $f_2(s,q)$ are uncertain polynomials as in (3.38) and for each $q \in Q$, $\deg[F_1(s,q)] \leq \deg[f_2(s,q)]$. An uncertain proper rational function, as in (3.39), is said to be *standard* if both $f_1(s,q)$ and $f_2(s,q)$ are standard uncertain polynomials.

Lemma 3.1 (see Reference [62] for proof)

Assume $f(s,q)$, as in (3.38), is an n-th order standard PHI polynomial and $f'(s,q)$ is an n'-th order standard uncertain polynomial satisfying the following conditions:

(1) $n' \leq n + 1$ and

(2) if $n' = n + 1$, then $\alpha_0'(q) > 0$ for all $q \in Q$.

Then, there exists an $\varepsilon^* > 0$ such that for any $\varepsilon \in (0,\varepsilon^*)$,

$$f_\varepsilon(s,q) = f(s,q) + \varepsilon f'(s,q)$$

is also PHI over Q.

The above lemmas can be easily generalized for uncertain polynomials with different degrees.

Lemma 3.2 (See [62] for proof.)

Assume $f(s,q)$, as in (3.38), is a n-th order standard PHI polynomial, and $f'(s,q)$ is a n'-th order standard uncertain polynomial satisfying the following conditions:

(1) $n' \leq n + 1$, and

(2) $\alpha_{n'}'(s,q) > 0$ for all $q \in Q$.

Then, there exists an $\varepsilon^* > 0$ such that for any $\varepsilon \in (0,\varepsilon^*)$,

$$f_\varepsilon(s,q) = sf(s,q) + \varepsilon f'(s,q)$$

is also PHI over Q.

<u>Lemma 3.3</u> (See Appendix D for proof.)

Assume $f(s,q)$, as in (3.38), is a standard PHI polynomial, and $f'(s,q)$ is a standard uncertain polynomial satisfying the following conditions: for each $q \in Q$,

(1) $d'(q) \leq d(q) + 1$, and

(2) if $d'(q) = d(q) + 1$ for some $q \in Q$, then $\alpha_0'(q) > 0$.

Then, there exists an $\varepsilon^* > 0$ such that for any $\varepsilon \in (0,\varepsilon^*)$,

$$f_\varepsilon(s,q) = f(s,q) + \varepsilon f'(s,q)$$

is also PHI over Q.

<u>Lemma 3.4</u> (See Appendix D for proof.)

Assume $f(s,q)$, as in (3.38), is a standard PHI polynomial and $f'(s,q)$ is a standard uncertain polynomial satisfying the following conditions:

(1) $d'(q) \leq d(q) + 1$ and

(2) $\alpha_{d'(q)}'(q) > 0$ for all $q \in Q$.

Then, there exists an $\varepsilon^* > 0$ such that for any $\varepsilon \in (0,\varepsilon^*)$,

$$f_\varepsilon(s,q) = sf(s,q) + \varepsilon f'(s,q)$$

is also PHI over Q.

<u>Robust Stabilizability of SISO Uncertain Systems</u>

Consider a SISO uncertain system represented by a family of transfer functions $\{P(s,q)\}$: $q \in Q$:

$$P(s,q) = \frac{N(s,q)}{D(s,q)} = \frac{\sum_{i=0}^{m(q)} \alpha_i(q) s^{m(q)-i}}{\sum_{i=0}^{m'(q)} \alpha_i'(q) s^{m'(q)-i}} \tag{3.40}$$

where $m(q)$ and $m'(q)$ are degree functions, $m(q)$ and $m'(q)$: $Q,Q \to \{0,1,2,\ldots\}$.

Definition 3.4

The family of SISO plants $\{P(s,q): q \in Q\}$ is said to be *simultaneously stabilizable* if there exists a rational compensator $C(s) = N_c(s)/D_c(s)$ having the following property: for each fixed $q \in Q$, the closed loop transfer function

$$P^*(s,q) = \frac{P(s,q)C(s)}{1 + P(s,q)C(s)}$$

has all its poles in the strict left half of the complex plane. In other words, the denominator polynomial of $P^*(s,q)$

$$\Delta(s,q) = N_c(s)N(s,q) + D_c(s)D(s,q) \tag{3.41}$$

is Hurwitz invariant over Q.

In Reference [60], the following assumptions play a central role for simultaneous stabilizability:

Assumption 3.1

Assume that the transfer function $P(s,q)$ satisfies the following conditions:

(1) Standardness: $P(s,q)$ is a standard uncertain rational function.

(2) One sign high frequency gain: The sign of the ratio of the leading coefficients of $D(s,q)$ and $N(s,q)$ is sign invariant over Q. In the sequel, we keep the sign of the leading coefficient of $D(s,q)$ positive.

(3) Minimum phase: For each fixed $q \in Q$, $P(s,q)$ is of minimum phase; i.e., all zeros of $P(s,q)$ lie in the strict left half plane.

Theorem 3.1 (see Reference [60] for proof)

Under the Assumptions 3.1, a SISO uncertain system $P(s,q)$ is simultaneously stabilizable. Furthermore, a robust stabilizer $C(s)$ can be constructed to be proper (or strictly proper, if desired) and stable.

Assumption 3.2

Assume that the transfer function $P(s,q)$ satisfies the following conditions:

(1) Standardness: $P(s,q)$ is a standard uncertain rational function.

(2) One sign high frequency gain): The sign of the ratio of the leading coefficients of $D(s,q)$ and $N(s,q)$ is sign invariant over Q.

(3) The numerator polynomial $N(s,q) = sN_1(s,q)$ and $N_1(s,q)$ is Hurwitz invariant over Q; i.e., all zeros of $N_1(s,q)$ lie in the strict left half plane.

(4) $\alpha_0'(q)\ \alpha_{m'(q)}'(q) > 0$ for all $q \in Q$.

Theorem 3.2 (See Appendix E for proof.)

Under the Assumptions 3.2, a SISO uncertain system $P(s,q)$ is simultaneously stabilizable. Furthermore, a robust stabilizer $C(s)$ can be constructed to be proper (or strictly proper, if desired) and stable.

Assumption 3.3

Assume that the transfer function $P(s,q)$ satisfies the following conditions:

(1) Standardness: $P(s,q)$ is a standard uncertain rational function.

(2) One sign high frequency gain: The sign of the ratio of the leading coefficients of $D(s,q)$ and $N(s,q)$ is sign invariant over Q.

(3) The denominator polynomial $D(s,q) = sD_1(s,q)$ and $D_1(s,q)$ is Hurwitz invariant over Q, i.e., all zeros of $D_1(s,q)$ lie in the strict left half plane.

(4) $\alpha_0(q)\alpha_{m(q)}(q) > 0$ for all $q \in Q$.

Theorem 3.3 (See Appendix F for proof.)

Under the Assumptions 3.3, a SISO uncertain system P(s,q) is simultaneously stabilizable. Furthermore, a robust stabilizer C(s) can be constructed to be proper (or strictly proper, if desired) and stable.

Assumption 3.4

Assume that the transfer function P(s,q) satisfies the condition that $P(s,q) = P_0(s) + P_1(s,q)$, where $P_0(s)$ is stable and $P_1(s,q)$ satisfies either Assumptions 3.1, 3.2, or 3.3, respectively.

Theorem 3.4 (See Appendix G for proof.)

Under the Assumptions 3.4, a SISO uncertain system P(s,q) is simultaneously stabilizable by a proper (or strictly proper, if desired) compensator.

The Stabilizing Compensator Synthesis and Illustrative Example

In this section, we extract the crucial ingredients for compensator construction from the proofs of Theorem 3.2, 3.3, and 3.4.

Procedure of Synthesis 3.1

Assume that the system satisfies Assumption 3.1. The compensator computation procedure is given in Reference [60].

Procedure of Synthesis 3.2

Assume that the system satisfies Assumption 3.2. Design compensator $C_1(s) = N_c(s)/D_{c,0}(s)$ by following Procedure 3.1 such that

$$\Delta_0(s,q) = N_c(s)N_1(s,q) + D_{c,0}(s)D(s,q)$$

is PHI over Q. Given $\Delta_i(s,q)$, select $\varepsilon_i > 0$ such that

$$\Delta_{i+1}(s,q) = s\Delta_i(s,q) + \varepsilon_i D_{c,0}(s)D(s,q)$$

is PHI and

$$D_{c,i+1}(s) = \varepsilon_0 s^i + \varepsilon_1 s^{i-1} + \ldots + \varepsilon_i$$

is Hurwitz. Assumption 3.2 guarantees the success of this procedure for $i = 0,1,2,\ldots,\lambda-1$. The compensator is given by

$$C(s) = N_c(s)/D_{c,\lambda}(s)D_{c,0}(s).$$

<u>Procedure of Synthesis 3.3</u>

Assume that the system satisfies Assumption 3.3. Set

$$\Delta_0(s,q) = D_1(s,q)$$

which is PHI over Q. Given $\Delta_i(s,q)$, select $\delta_i > 0$ such that

$$\Delta_{i+1}(s,q) = s\Delta_i(s,q) + \delta_i N(s,q)$$

is PHI. Assumption 3.3 guarantees the success of this procedure for $i = 0,1,2,\ldots,\lambda-1$. The numerator of the compensator is given by

$$N_c(s) = \delta_0 s^{-1} + \delta_1 s^{-2} + \ldots + \delta_{\lambda-1}.$$

Set

$$\Delta_1'(s) = N_c(s)N(s,q) + s^\lambda D(s,q)$$

For given $\Delta_i(s,q)$, select $\varepsilon_i > 0$ such that

$$\Delta_{i+1}'(s,q) = \Delta_i'(s,q) + \varepsilon_i s^{\lambda+1} D(s,q)$$

is PHI and

$$D_{c,i}(s) = q + \varepsilon_1 s^1 + \ldots + \varepsilon_i s^i$$

is Hurwitz. Assumption 3.3 guarantees the success of this procedure for $i = 1,2,\ldots,l \geq \lambda-1$. The compensator is given by

$$C(s) = N_c(s)/D_{c,l}(s).$$

Procedure of Synthesis 3.4

Assume that the system satisfies Assumption 3.4. Design a compensator $C_1(s) = N_{c,1}(s)/D_{c,1}(s)$ such that

$$\Delta_0(s,q) = N_{c,1}(s)N_1(s,q) + D_{c,1}(s)D_1(s,q)$$

is PHI over Q. Then the desired compensator C(s) is given by

$$C(s) = \frac{C_1(s)}{1 - C_1(s)P_0(s)}$$

In the computation procedures, it is often required to determine Hurwitz invariance of uncertain polynomials. Numerous methods can be found in References [2] through [5], as well as the results of subsection 2.2.

Example 3.4

A continuum of plants is described by

$$P(s,q) = P_0(s) + P_1(s,q) = \frac{1}{s+1} + \frac{s(s+q_1)}{s^2 - q_2 s + 1}, \quad q_1 \in [1,2], \quad q_2 \in [-2,2].$$

Since Assumption 3.4 is observed to be satisfied by inspection, this family of plants is simultaneously stabilizable. We now construct a stabilizing compensator. Design $C_0(s) = N_0(s)/D_0(s)$ such that

$$\Delta_0(s,q) = N_0(s)(s + q_1) + (D_0(s)(s^2 - q_2 s + 1))$$

is Hurwitz invariant. Following Procedure 3.1, we take $N_0(s) = 1$ and $D_0(s) = \varepsilon_0$; then

$$\Delta_0(s,q) = (s + q_1) + \varepsilon_0(s^2 - q_2 s + 1) = \varepsilon_0 s^2 + (1 - \varepsilon_0 q_2)s + \varepsilon_0 + q_1$$

Obviously, $\varepsilon_0 < 0.5$ will make $\Delta_0(s,q)$ Hurwitz invariant, so we choose $\varepsilon_0 = 0.4$. Now consider

$$\Delta_1(s,q) = s\Delta_0(s,q) + \varepsilon_1(s^2 - q_2 s + 1)$$

It is easy to check (using the Kharitonov test [2], for instance) that when $\varepsilon_1 = 0.4$, $\Delta_1(s,q)$ will be Hurwitz invariant. Hence, the robust compensator for $P_1(s,q)$ is

$$C_1(s) = \frac{1}{0.4s + 0.4} = \frac{2.5}{s + 1}$$

Consequently, the robust compensator $C(s)$ for $P(s,q)$ is given by

$$C(s) = \frac{C_1(s)}{1 - C_1(s)P_0(s)} = \frac{2.5(s + 1)}{s^2 + 2s - 1.5}$$

IV. ROBUST CONTROL OF LINEAR SYSTEMS WITH COMBINED STRUCTURED (PARAMETRIC) AND UNSTRUCTURED (UNMODELED DYNAMICS) UNCERTAINTY

Most of the published literature on the robust stabilization (and performance) problem treats the two cases of structured uncertainty (parametric variations) and unstructured uncertainty (unmodeled dynamics) separately. The aspect of unstructured uncertainty is conveniently handled in the frequency domain framework, whereas the structured uncertainty problem is treated, as discussed in the previous chapters, both in frequency domain and time domain frameworks. While research in each of these categories is still growing at an impressive pace, with a variety of problems being addressed and resolved, the motivation for carrying out research that treats the combined case in a unifying framework comes from a variety of reasons. Many engineering applications naturally involve the presence of both types of uncertainties; the application of large space structure control is a case in point. Also, the mathematical frameworks considered for these two cases of uncertainty cannot always be transformed from one to the other. Hence, in this final leg of the research work done under this contract, we consider the combined uncertainty problem. In Section 4.1, we concentrate on the Large Space Structure models in the state space framework and model the unmodeled dynamics error as an additive perturbation, and using the theory developed by the author, obtain bounds on the control gains to keep the closed loop system stable under both control and observation spillover. Then the aspect of parameter variations is addressed and, using the same tool as before, a robust

controller is designed for stabilization under these parameter variations. Then in section 4.2 a frequency domain based design method is given to construct a single compensator for robust stabilization in the presence of combined structured and unstructured uncertainties. Apart from the difference in perspectives, another main difference between these two methods is that in the time domain design, the two types of uncertainties are accommodated in sequence, whereas in the frequency domain method, simultaneous presence is addressed.

4.1 TIME DOMAIN (STATE SPACE) DESIGN METHOD FOR LSS MODELS

One fundamental design problem in the control of LSS is the control of a large dimensional system with a controller of much smaller dimension. In this framework, the modeling error is "mode truncation." One prominent effect of mode truncation in the control design process is the instability caused due to the interaction with residual modes, labeled "spillover" [63]. Furthermore, the parameters of the control design model itself (such as modal frequencies, mode shapes, and damping ratios) are all known imprecisely. For instance, the errors associated with the finite element models of LSS increase typically with increase in the mode number (i.e., with model frequency). Even though in most cases the effect of this "parameter variation" error is performance degradation, it is sometimes possible to encounter instability also. In the light of these observations it is evident that "robustness" is an extremely desirable feature of any active control design proposed for LSS control.

The bulk of the published literature on the "robust stability" of LSS models addresses the modeling error "mode truncation" (i.e., the use of reduced order models/controllers) in the design [64]-[68]. The problem of

"parameter errors" in the control design of LSS has not received as much attention as the spillover problem. In the former category, recent results using the frequency domain viewpoint treat the mode truncation error as an unstructured (high frequency) uncertainty and propose solution methods such as LQG/LTR [69]-[70]. Applications of LQG/LTR method in LSS control problems are discussed in References [71] and [72]. In the time domain viewpoint, Ikeda and Siljak [73] give bounds on the spillover terms using vector Lyapunov functions. Time domain control design schemes which treat both mode truncation and parameter variations for LSS control have been relatively scarce [74]-[75], and even among these, Reference [74] treats only the small parameter variation case while Reference [75] assumes stability and addresses the performance aspect. The proposed method is a contribution in this direction. In this section both the "mode truncation error" and the "parameter error" are treated as an additive perturbation, and the stability robustness analysis for structured uncertainty developed by the author in References [77] and [20] is extended to treat the above problem to obtain bounds on the "spillover" terms to maintain stability. In this context, the methodology of this paper is similar in spirit to the paper by Ikeda and Siljak, but with considerable difference in detail. One important feature of the proposed method is that the bounds on control/estimator gains can be determined apriori, simply using the open loop modal data; also, the tradeoff between these bounds and the number of modes one wishes to control is clearly brought out.

Description of the Proposed Approach

As noted in the previous section, the two major design issues of concern in LSS control are those of the avoidance of spillover [65], and parameter

uncertainty. Spillover is brought about by a "two-model" design philosophy in which a higher order "evaluation model" (which serves the role of the physical system in system evaluation) is driven by a lower order controller of sufficiently large control effort. Parameter uncertainty arises because of the inherent inaccuracies in the development of mathematical models (e.g. finite element modeling). In the proposed design scheme, both of these issues are addressed in a systematic way. First the aspect of mode truncation error (spillover) is addressed. While there are many methods available for obtaining the reduced order controller [75]-[76], we employ the route of reducing the open loop evaluation model (by some known criterion) and then building a full order linear quadratic gaussian (LQG) controller for the reduced order model. We then apply the results of the Elemental Perturbation Bound Analysis (EPBA) discussed in section 3.1 to the closed loop system model for obtaining upper bounds on the control/estimator gains as a function of the number of modes retained in the reduced order control design model to guarantee avoidance of spillover [78]. Then, armed with these bounds on the gains, at the control design stage (on the reduced order model), once again the results of EPBA are used to obtain upper bounds on the tolerable variations in the control design model parameters to maintain stability [79]. Thus the proposed "robust controller" is that controller which possesses as high a tolerable perturbation in the system parameters as possible along with the guarantee of avoidance of spillover instability. The important features of the proposed controller are that (1) the control/estimator gain bounds are calculated, apriori, using open loop modal data; (2) that it is a practically implementable estimator based control law; and most importantly, (3) that it brings about the tradeoff between the nominal performance attainable and the

number of modes included in the control design model (i.e., the dimension of the reduced order model).

<u>LSS Models and Mode Truncation as Perturbation</u>:

Consider the standard state space description of an LSS evaluation model with N elastic modes:

$$\begin{aligned} \dot{x} &= Ax + Bu + Dw \qquad x(0) = x_o; \quad x \to R^{n-2N}, \quad u \to R^m \\ y &= Cx \qquad\qquad\qquad\qquad\qquad\quad y \to R^k \\ z &= Mx + v \qquad\qquad\qquad\qquad\qquad\quad z \to R^{\ell} \end{aligned} \tag{4.1a}$$

where

$$x^T = [x_1^T, x_2^T, \ldots, x_N^T]; \quad x_i = \begin{bmatrix} \eta_i \\ \dot{\eta}_i \end{bmatrix} \tag{4.1b}$$

$$A = \text{Block diag. } [\ldots A_{ii} \ldots], \quad A_{ii} = \begin{bmatrix} 0 & 1 \\ -\omega_i^2 & -2\zeta\omega_i \end{bmatrix} \tag{4.1c}$$

$$B^T = [B_1^T, B_2^T, \ldots, B_N^T]; \quad B_i = \begin{bmatrix} 0 \\ b_i^T \end{bmatrix} \tag{4.1d}$$

$$D^T = [D_1^T, D_2^T, \ldots, D_N^T]; \quad D_i = \begin{bmatrix} 0 \\ d_i^T \end{bmatrix} \tag{4.1e}$$

$$C = [C_1, C_2, \ldots, C_N] \quad \text{and} \quad M = [M_1, M_2, \ldots, M_N] \tag{4.1f}$$

$$\xi \begin{bmatrix} w(t) \\ v(t) \end{bmatrix} [w^T(\tau) v^T(\tau)] = \begin{bmatrix} W & 0 \\ 0 & \rho_e V_0 \end{bmatrix} \tilde{\delta}(t - \tau) \tag{4.1g}$$

Here y is the vector of variables we wish to control, z is the measurement vector, w and u are zero mean white noise processes with constant covariances W and $\rho_e V_o$ respectively, where $\rho_e > 0$ is taken as a design variable and ξ is the expectation operator and $\bar{\delta}$ is the Dirac delta function.

Assuming vibration suppression of the flexible structure to be the control objective, a quadratic performance index may be written as

$$J = \lim_{t\to\infty} \frac{1}{t} \xi \int_0^t \left[\left(\sum_{i=1}^{N} \omega_i^2 \eta_i^2 + \dot{\eta}_i^2 \right) + \rho_c u^T u \right] dt \tag{4.2}$$

which can be written in the form

$$J = \lim_{t\to\infty} \frac{1}{t} \xi \int_0^t (y^T y + \rho_c u^T u)\, dt = \lim_{t\to\infty} \frac{1}{t} \xi \int_0^t (y^T y + \rho_c u^T u)\, dt$$

$$= J_y + \rho_c J_u \tag{4.3}$$

where the matrix C of (4.1) is given by

$$C = \text{Block diag.} \quad [\ldots C_i \ldots] \tag{4.4a}$$

and

$$C_i = \begin{bmatrix} \omega_i & 0 \\ 0 & 1 \end{bmatrix} \tag{4.4b}$$

The scalar $\rho_c > 0$ is also taken as a design variable.

Assuming the modes η_i are ordered in increasing order of frequency, let us retain the first N_r modes for control design purposes. Accordingly, the reduced order control design model of dimension $n_R (= 2N_r) < n$ is given by

$$\dot{x}_R = A_R x_R + B_R u + D_R w \qquad x_R \to R^{n_R} \tag{4.5}$$

$$y_R = C_R x_R \qquad z_R = M_R x_R + v$$

where the above control design model is obtained by direct truncation of the full order model given by

$$\dot{x} = \begin{bmatrix} \dot{x}_R \\ \dot{x}_T \end{bmatrix} = \begin{bmatrix} A_R & 0 \\ 0 & A_T \end{bmatrix} \begin{bmatrix} x_R \\ x_T \end{bmatrix} + \begin{bmatrix} B_R \\ B_T \end{bmatrix} u + \begin{bmatrix} D_R \\ D_T \end{bmatrix} w$$

$$y = [C_R \quad C_T] \begin{bmatrix} x_R \\ x_T \end{bmatrix} \qquad z = [M_R \quad M_T] \begin{bmatrix} x_R \\ x_T \end{bmatrix} \tag{4.6}$$

Let the full order optimal control for the reduced order model be given by

$$u = G_R \hat{x}_R$$

$$\dot{\hat{x}}_R = A_R \hat{x}_R + B_R u + F_R(z - M_R \hat{x}_R) \tag{4.7}$$

$$= \hat{A}_R \hat{x}_R + F_R z \text{ where } \hat{A}_R = A_R + B_R G_R - F_R M_R$$

where G_R and F_R are the standard "controller" and "estimator" gain matrices obtained by minimizing the performance index

$$J_R = \lim_{t \to \infty} \frac{1}{t} \xi \int_0^t \left[\left(\sum_{i=1}^{N_r} \omega_i^2 \eta_i^2 + \dot{\eta}_i^2 \right) + \rho_c u^T u \right] dt \tag{4.8a}$$

$$= \lim_{t \to \infty} \frac{1}{t} \xi \int_0^t (x_R^T C_R^T C_R x_R + \rho_c u^T u)\, dt = J_{yR} + \rho_c J_u \tag{4.8b}$$

and are given by

$$G_R = -\frac{1}{\rho_c} B_R^T K_c \tag{4.9a}$$

$$K_c A_R + A_R^T K_c - \frac{K_c B_R B_R^T K_c}{\rho_c} + C_R^T C_R = 0 \tag{4.9b}$$

and

$$F_R = \frac{1}{\rho_c} K_e M_R^T V_o^{-1} \tag{4.9c}$$

$$K_c A_R + A_R^T K_e - K_e M_R^T \frac{V_o^{-1}}{\rho_e} M_R K_e + D_R W D_R^T = 0 \tag{4.9d}$$

The closed loop system matrix for the control design model given by

$$\bar{A}_R = \begin{bmatrix} A_R & B_R G_R \\ F_R M_R & \hat{A}_R \end{bmatrix} \tag{4.10}$$

is asymptotically stable under the usual assumptions of controllability and observability.

The closed loop system for the evaluation model is obtained by forcing the evaluation model with the controller of the control design model. Thus, we have

$$\begin{bmatrix} \dot{x}_R \\ \dot{\hat{x}}_R \\ \dot{x}_T \end{bmatrix} = \begin{bmatrix} A_R & B_R G_R & 0 \\ F_R M_R & \hat{A}_R & F_R M_T \\ 0 & B_T G_R & A_T \end{bmatrix} \begin{bmatrix} x_R \\ \hat{x}_R \\ x_T \end{bmatrix} + \begin{bmatrix} D_R & 0 \\ 0 & F_R \\ D_T & 0 \end{bmatrix} \begin{bmatrix} w \\ \\ v \end{bmatrix} \tag{4.11a}$$

$$\begin{bmatrix} y \\ u \end{bmatrix} = \begin{bmatrix} C_R & 0 & C_T \\ 0 & G_R & 0 \end{bmatrix} [\, x_R \quad \hat{x}_R \quad x_T \,]^T$$

i.e.,

$$\dot{X}_b = A_b X_b + D_b \bar{w}, \quad y_b = C_b X_b \tag{4.11b}$$

where the stability of the matrix A_b is dictated by the spillover terms $B_T G_R$ and $F_R M_T$.

We now write

$$A_b = \bar{A} + E_b = \begin{bmatrix} A_R & 0 & 0 \\ 0 & A_R & 0 \\ 0 & 0 & A_T \end{bmatrix} + \begin{bmatrix} 0 & B_R G_R & 0 \\ F_R M_R & B_R G_R - F_R M_R & F_R M_T \\ 0 & B_T G_R & 0 \end{bmatrix} \tag{4.12}$$

Note that the E_b matrix contains the spillover terms $F_R M_T$ and $B_T G_R$ and that the $\bar{A}$ matrix is asymptotically stable with block diagonal structure. In other words, the spillover terms are treated as an additive perturbation. Let us denote the matrix E_b as the "spillover matrix."

It is to be noted that the matrix E_b is not known prior to the control design and that $\bar{A}$ is essentially the open loop plant matrix.

Let

$$\bar{f} = \mathrm{Max}(|F_{Rij}|), \quad \bar{g} = \mathrm{Max}(|G_{Rij}|) \tag{4.13}$$

$$\bar{b} = \mathrm{Max}(|B_{ij}|), \quad \bar{m} = \mathrm{Max}(|M_{ij}|) \tag{4.14}$$

and

$$\bar{h} = \mathrm{Max}(\bar{f}, \bar{g})$$

Then the maximum element (in the worst case situation) in the matrix E_b will be

$$\epsilon = \bar{h}(m\bar{b} + \ell\bar{m}) = \bar{h}\,\bar{c} \tag{4.15}$$

where the constant $\bar{c}$ can be computed, apriori, from the given model. We thus write the "majorant" of E_b as

$$\Delta_b = \epsilon U \tag{4.16}$$

where

$$U = \begin{bmatrix} 0 & U_{e1} & 0 \\ U_{e2} & U_{e3} & U_{e4} \\ 0 & U_{e5} & 0 \end{bmatrix} \tag{4.17}$$

with $U_{e1ij} = 1$ whenever there is a nonzero entry in the $B_R G_R$ matrix and $U_{e1ij} = 0$ for all zero entries and so on for the other matrices $U_{e2}, \ldots, U_{e5}$.

Now using the robust stability result of subsection 3.1 (specifically, equation (3.3)), we conclude that the closed loop system matrix A_b is stable if

$$\varepsilon = \bar{h}\,\bar{c} < \frac{1}{\sigma_{max}(PU)_s} = \mu_R \tag{4.18a}$$

where P satisfies

$$P\bar{A} + \bar{A}^T P + 2I = 0 \tag{4.18b}$$

Note that the above equation can be easily solved exploiting the structure of the $\bar{A}$ matrix. Accordingly,

$$P = \text{Block diag. } [\ldots P_{ii} \ldots] \tag{4.18c}$$

$$P_{ii} = \begin{bmatrix} \left(\dfrac{1 + 4\zeta^2}{2\zeta\omega_i} + \dfrac{\omega_i}{2\zeta}\right) & 1/\omega_i^2 \\ \dfrac{1}{\omega_i^2} & \dfrac{1 + \omega_i^2}{2\zeta\omega_i^3} \end{bmatrix} \tag{4.18d}$$

Note that in this case $P_{iim} = P_{ii}$ (i.e., all the elements of P_{ii} are positive).

Denoting $\mu_{R\alpha}$ as the bound obtained as in (4.18a) when A_R contains α modes ($\alpha = 1,2,\ldots,N-1$), we can get the bound on the control/estimator gain as a function of the number of controlled modes, as

$$\bar{h}_\alpha < \frac{\mu_{R\alpha}}{\bar{c}} = h_{m\alpha} \tag{4.19}$$

Thus any controller/estimator gains (G_R and F_R where G_R is $m \times 2\alpha$ and F_R is $2\alpha \times \ell$) such that

$$|G_{Rij}| < h_{m\alpha}, \qquad |F_{Rij}| < h_{m\alpha} \tag{4.20}$$

would keep the closed loop system matrix A_b stable (i.e., avoid spillover instability).

<u>Remark</u>: The most important point to note is that the bound $h_{m\alpha}$ is obtained <u>apriori</u> before control design by simply utilizing the open loop modal data such as the modal frequency, mode shape slopes at actuator/sensor locations (B and M matrices), open loop modal damping ratio δ, the number of control inputs, the number of measurements, and the number of modes included in the control design model.

Since with gains satisfying (4.20) (obtained by design variables ρ_c and ρ_e) the closed loop system matrix A_b is stable, one can proceed to compute the nominal output regulation cost J_y and the corresponding control effort J_u and obtain a plot of J_y vs. J_u for each $\alpha = 1,2,\ldots,N-1$.

<u>Stability of the Control Design Model under Parameter Uncertainty</u>

Once the stability under mode truncation is taken care of, the next issue is the stability of the control design model itself. From the control design equations of (4.8)-(4.20), it is seen that under the nominal situation the closed loop reduced order control design model is asymptotically stable. However, in an LSS control problem the parameters of the plant matrix A_R, namely, the modal frequencies and dampings as well as the mode shape slopes at actuator (B_R) and sensor (M_R) locations are known to be uncertain. It is also known that the uncertainty in these parameters tends to increase with an

increase in the mode number. Thus with variations ΔA_R, ΔB_R, ΔM_R, etc., the control/estimator set obtained in the previous section cannot guarantee stability of the reduced order closed loop system.

Clearly the appropriate step then is to pick that set of control/estimator gains which not only satisfy condition (4.20), but also maximize the allowable range of variations in ΔA_R, ΔB_R, and ΔM_R for robust stability. Towards this direction, the stability robustness index β_{SR} of (3.7) or (3.8) can be used as a robustness measure to be maximized with parameter uncertainty treated as an additive perturbation. In doing so, one can even utilize the uncertainty structure, namely the fact that the uncertainty in modal parameters increases with increase in mode number. One way of characterizing this structure is as follows (here we employ the relative variation format of (3.5)):

$$\Delta A = \delta_a \begin{bmatrix} 0 & 0 & & & & & \\ \otimes_1 & \otimes_1 & & & & & \\ & & 0 & 0 & & & \\ & & 2\otimes_2 & 2\otimes_2 & & & \\ & & & & 0 & 0 & \\ & & & & 3\otimes_3 & 3\otimes_3 & \\ & & & & & & \ddots \end{bmatrix}$$

$$\Delta B = \delta_b \begin{bmatrix} 0 \\ b_1^T \\ 0 \\ 2b_2^T \\ 0 \\ 3b_3^T \\ \vdots \end{bmatrix} \tag{4.21}$$

(And similarly for ΔM_R)

where $\otimes_i$ indicate the nominal entries corresponding to the i-th mode. We assume $\delta_a = \delta_b$, which are not known.

With the above uncertainty structure, again using the result for robust stability of (3.3), one can get a bound on δ_a as

$$\delta_a < \mu_s \tag{4.22}$$

where μ_s is obtained as usual by solving a Lyapunov equation of the type (4.5) using the nominally asymptotically stable closed loop system matrix $\bar{A}_R$. Note that μ_s is a function of the control/estimator gains. We can thus define μ_s as the robustness measure for parameter uncertainty, i.e.,

$$\beta_{SR} = \mu_s \tag{4.23}$$

and plot β_{SR} vs. the control effort J_u.

Finally the proposed robust controller, for a given $\alpha = 1,2,\ldots,N-1$ is that controller which satisfies (1) condition (4.29), (2) $J_y \leq J_{ym}$ (maximum tolerable output regulation cost), and (3) as high β_{SR} as possible.

Of course if there is no stringent requirement on J_y, then one can select a gain set that achieves maximum β_{SR}.

4.2 FREQUENCY DOMAIN (TRANSFER FUNCTION) DESIGN METHOD

In this section, we consider the simultaneous presence of plant uncertainties characterized by low order parameter (structured) variation and high order (unstructured) uncertainty for systems described by transfer functions. When the system uncertainty is mainly caused by high order unstructured uncertainty, the most popular design model is characterized by a given fixed nominal plant with a set of norm-bounded perturbations. If perturbation is unknown, various optimization methods (References [80]-[83])

are applied to design a feedback control for the nominal plant so that the closed loop system can tolerate "maximum" perturbation. If perturbation bound norms are given, closed loop stability conditions are derived by Doyle and Stein [69] and by Chen and Desoer [84] based on a generalized version of the Nyquist stability theorem. Combining Nevanlinna-Pick theory and Youla's parametrization, robust stabilizability conditions are provided by Kimura [85] for single-input single-output (SISO) systems and are further generalized by Vidyasagar and Kimura [86] for multi-input multi-output (MIMO) systems.

On the other hand, if parameter variation is of major concern, then the concept of "simultaneous stabilization" as discussed in section 3.2 becomes the appropriate method of attack.

As mentioned in the beginning of this section, most of the published literature treats these two major kinds of uncertainties separately. The combined presence naturally raises a question: given an uncertain system having both parameter variation and unstructured uncertainty, is it possible to find a robust control which guarantees the closed loop stability over the entire perturbation range? If so, how to design a desired compensator? This section intends to examine this problem. A new design model is proposed to characterize an uncertain system containing those two different uncertainties. Then stability criteria for closed loop systems are derived from classical results for this new model. It is then found that under certain conditions an uncertain system represented by the new model can be stabilized by a proper (or strictly proper if desired) and stable compensator. A procedure of synthesizing the robust stabilizer is also provided. All proofs are in the appendices.

Preliminary Definitions and Lemmas:

In addition to the definitions and lemmas introduced in section 3.2, we need the following lemmas:

Lemma 4.1 (See Appendix H for proof.)

Assume $f(s,q)$, as in (3.38), is an n-th order standard positive Hurwitz invariant uncertain polynomial and

$$|f(j\omega,q)|^2 \Delta g(\omega^2,q) = \beta_0(q)\omega^{2n} + \beta_1(q)\omega^{2n-2} + \ldots + \beta_n(q), \tag{4.25}$$

then

(1) $\min\{\beta_0(q): q \in Q\} = b_0 > 0$; and (4.26a)

(2) $\min\{g(\omega^2,q): q \in Q, \omega \geq 0\} = k > 0$. (4.26b)

Lemma 4.2 (See Appendix I for proof.):

Assume $f(s,q)$, as in (3.38), is a standard positive Hurwitz invariant (PHI) polynomial with degree $d(q)$ and $f'(s,q)$ is a standard uncertain polynomial with degree $d'(q)$ and leading coefficient $\alpha_0'(q)$ satisfying the following conditions: For all $q \in Q$,

(1) $d'(q) \leq d(q) + 1$, and

(2) if $d'(q) = d(q) + 1$, then $\alpha_0'(q) > 0$.

Then, given any constant $a > 1$, there exists an $\varepsilon^* > 0$ such that for any $\varepsilon \in (0,\varepsilon^*)$,

$$f_\varepsilon(s,q) = f(s,q) + \varepsilon f'(s,q) \tag{4.27a}$$

is also a standard PHI polynomial and

$$\frac{|f(j\omega,q)|}{|f_\varepsilon(j\omega,q)|} < a \tag{4.27b}$$

for all $q \in Q$ and $\omega \geq 0$.

Modeling of Uncertain Systems:

In the recent literature on robust stability theory, various mathematical models of uncertain linear systems are proposed in which, however, the two most widely used models are the following:

Model 1 (Low Order Parameter Variation Model):

In the frequency domain, an uncertain system is represented by a family of rational matrices $\{P(s,q): q \in Q\}$ where Q is a prespecified index set, for each $q \in Q$, $P(s,q)$ is an $n \times m$ rational matrix and the highest order of $P(s,q)$ is a given finite number.

In the state space counterpart, an uncertain system is represented by a family of matrices $\{A(q),B(q),C(q),D(q): q \in Q\}$ in the state space equations

$$\begin{aligned} x &= A(q)x + B(q)U, \\ y &= C(q)x + D(q)u, \quad q \in Q, \end{aligned} \tag{4.28}$$

where Q is a prespecified set and the dimensions of $A(q)$, $B(q)$, $C(q)$, and $D(q)$ are given finite numbers.

Model 2 (High Order Norm Bounded Uncertainty Model):

An uncertain system is represented by a class of plants [80]-[85]

$$A(P_0(s),r(s)) \triangleq \{P_0(s) + L(s): \|L(j\omega)\| < |r(j\omega)|;\ \forall\ \omega \geq 0\} \tag{4.29a}$$

or

$$M(P_0(s),r(s)) \triangleq \{(I + L(s))P_0(s): \|L(j\omega)\| < |r(j\omega)|;\ \forall\ \omega \geq 0\} \tag{4.29b}$$

where $P_0(s)$ is a given nominal plant, $r(s)$ is a prespecified rational function and $L(s)$ is an unknown but stable rational matrix whose norm is bounded by $|r(j\omega)|$, $A(\cdot)$ denotes additive model and $M(\cdot)$ denotes multiplicative model.

An important, but less recognized fact is that an uncertain system represented by Model 1 may not always be represented by Model 2 and vice versa. It may be easy to see that Model 2 allows infinite order uncertainty

to be contained while Model 1 cannot; it can also be seen that an uncertain system represented by Model 1 may not be always represented by Model 2. For example, consider an uncertain system described by

$$P(s,q),\ q \in \{1,2\}$$

in which

$$P(s,1) = \frac{1}{s - 1}, \text{ and}$$

$$P(s,2) = \frac{1}{s^2 - 2s - 2}.$$

Obviously, Model 2 is not suitable for describing this family of systems.

Based on the observations above, to characterize a plant containing both parameter variation and high frequency unstructured uncertainty, we propose a new model of uncertain systems combining both Model 1 and Model 2 for studying robust stability problems.

<u>Definition 4.1</u>:

An *uncertain system* $A(P_0(s,q),r(s),Q)$ is defined by

$$A(P_0(s,q),r(s),Q) \triangleq \{P_0(s,q) + L(s):\ q \in Q;\ \|L(j\omega)\| < |r(j\omega)|,\ \forall\ \omega \geq 0\} \tag{4.30}$$

and an *uncertain system* $M(P_0(s,q),r(s),Q)$ is defined by

$$M(P_0(s,q),r(s),Q) \triangleq \{(I + L(s))P_0(s,q):q \in Q;\|L(j\omega)\| < |r(j\omega)|,\ \forall\ \omega \geq 0\} \tag{4.31}$$

where Q is a prespecified index set; for each $q \in Q$, the nominal plant $P_0(s,q)$ is an $n \times m$ rational uncertain matrix; $r(s)$ is a prespecified rational function and $L(s)$ is any unknown but <u>stable</u> rational matrix whose norm is bounded by $|r(j\omega)|$.

<u>Stability Criteria</u>:

In the previous section, we proposed a new model of uncertain systems. Our main objective is to design a robust feedback control to stabilize an

uncertain system represented by the new model. To this end, we need stability criteria for the closed loop systems. First, we introduce a standard result for uncertain systems with only unstructured uncertainty [86].

Lemma 4.3

If a controller $C(s)$ stabilizes $P_0(s)$, then $C(s)$ stabilizes all plants in the class $A(P_0(s), r(s))$ if and only if

$$\|C(s)(I + P_0(s)C(s))^{-1}r(s)\|_\infty \leq 1.$$

This lemma can be naturally extended for our new model; no proof is needed.

Theorem 4.1

If $C(s)$ stabilizes $P_0(s,q)$, then $C(s)$ stabilizes all plants in the class $A(P_0(s,q),r(s),Q)$ if and only if

$$\|C(s)(I + P_0(s,q)C(s))^{-1}r(s)\|_\infty \leq 1$$

for all $q \in Q$.

For SISO systems, a straightforward computation directly leads to the following corollary.

Corollary 4.1

A controller $C(s) \Delta N_c(s)/D_c(s)$ stabilizes a SISO uncertain system $A(P_0(s,q),r(s),Q)$ with $P_0(s,q) \Delta N(s,q)/D(s,q)$ if

$$\Delta(s,q) \Delta D_c(s)D(s,q) + N_c(s)N(s,q) \tag{4.32}$$

is Hurwitz invariant over Q and

$$\frac{|N_c(j\omega)N(j\omega,q)|\ |r(j\omega)|}{|D_c(j\omega)D(j\omega,q) + N_c(j\omega)N(j\omega,q)|} \leq 1 \tag{4.33}$$

for all $q \in Q$ and $\omega \geq 0$.

Robust Stabilizability of SISO Uncertain Systems:

Consider a SISO uncertain system $A(P_0(s,q),r(s),Q)$ or $M(P_0(s,q),r(s),Q)$ as in Definition 4.1, where

$$P_0(s,q) \triangleq \frac{N(s,q)}{D(s,q)} \triangleq \frac{\sum_{i=0}^{m(q)} \alpha_i(q) s^{m(q)-i}}{\sum_{i=0}^{m'(q)} \alpha_i'(q) s^{m'(q)-i}}$$

where the symbols have the same definitions as in (3.40). The following assumptions play a central role for robust stabilizability.

<u>Assumption 4.1</u> (Standardness)

$P_0(s,q)$ is a standard uncertain rational function.

<u>Assumption 4.2</u> (Minimum Phase)

For each fixed $q \in Q$, $P_0(s,q)$ is of minimum phase; i.e., all zeros of $P_0(s,q)$ lie in the strict left half plane.

<u>Assumption 4.3</u> (One Sign High Frequency Gain)

The sign of the ratio of the leading coefficients of $D(s,q)$ and $N(s,q)$ is sign invariant over Q. In the sequel, we keep the sign of the leading coefficient of $D(s,q)$ positive.

<u>Assumption 4.4</u> (Boundedness)

For an uncertain system $\mathbf{A}(P_0(s,q),r(s),Q)$,

$$\sup \left\{ \frac{|r(j\omega)|}{|P_0(j\omega,q)|} : \quad q \in Q;\ \omega \geq 0 \right\} < 1.$$

For an uncertain system $\mathbf{M}(P_0(s,q),r(s),Q)$,

$$\sup\{ |r(j\omega)| : \quad \omega \geq 0\} < 1.$$

<u>Remark</u>: An uncertain system satisfying Assumptions 4.1-4.4 can be viewed as a family of finite or infinite number of plants. Each plant may have a different nominal plant with a different relative degree and unknown but norm bounded stable high frequency uncertainty. The zeros of each nominal

plant are in the strict left half plane. There is no restriction on pole positions of the nominal plants.

Theorem 4.2 (See Appendix J for proof.)

Under the Assumptions 4.1-4.4, a SISO uncertain system $A(P_0(s,q),r(s),Q)$ or $M(P_0(s,q),r(s),Q)$ is robust stabilizable. Furthermore, a robust stabilizer $C(s)$ can be constructed to be proper (or strictly proper if desired) and stable. The minimum order of $C(s)$ is equal to

$k = \max\{m'(q) - m(q): q \in Q\} - 1$ or $k = 0$ when $\max\{m'(q) - m(q): q \in Q\} = 0$.

Remark: When high frequency unstructured uncertainty is neglected, that is, $L(s) = 0$, Theorem 4.2 coincides with the main result in Reference [60]. When parameter variations are not under consideration, i.e., $P_0(s,q)$ reduces to $P_0(s)$, it is of interest to compare Theorem 4.2 with some existing results, such as References [69] and [85]. First, both references require a full order controller, while Theorem 4.2 provides a significantly lower order compensator, depending only on the largest relative degree of all nominal plants. Secondly, the stability margin for a system $M(P_0(s),r(s),Q)$ with a minimum phase nominal plant is improved to $\inf|r(j\omega)| < 1$ from $|r(j\omega)| < 1/2$ obtained by applying LQG/LTR in Reference [69]. Finally, in Theorem 4.2 there are no special restrictions on the relative degree of $r(s)$ and the locations of unstable poles of $P_0(s)$, as in Reference [85].

The Stabilizing Compensator Synthesis:

In this section, we extract the crucial ingredients for compensator construction from the proof of Theorem 4.2.

Procedure of Synthesis 4.1

Step 1: Consider a compensator of the form

$$C(s) \triangleq \frac{N_c(s)}{D_c(s)} \triangleq \frac{\lambda_k s^k + \lambda_{k-1}s^{k-1} + \ldots + \lambda_1 s + \lambda_0}{\varepsilon_l s^l + \varepsilon_{l-1}s^{l-1} + \ldots + \varepsilon_1 s + \varepsilon_0} \; ; \quad l \geq k. \tag{4.34}$$

Choose $N_c(s)$ to be <u>any</u> Hurwitz polynomial with $\lambda_k > 0$ when $N(s,q)$ is PHI or $\lambda_k < 0$ when $N(s,q)$ is negative Hurwitz invariant and degree

$$k \geq \max\{m'(q) - m(q): q \in Q\} - 1$$

or $k = 0$ when $\max\{m'(q) - m(q): q \in Q\} = 0$.

<u>Step 2</u> (Initialization): Define

$\Delta_0(s,q) \triangleq N_c(s)N(s,q)$;

$D_{c,0}(s) \triangleq 0$;

$$\delta = \inf\left\{ \frac{|P_0(j\omega,q)|}{|r(j\omega)|} : q \in Q;\ \omega \geq 0 \right\} - 1$$

for an uncertain system $A(P_0(s,q),r(s),Q)$ and

$$\delta = \inf\left\{ \frac{1}{|r(j\omega)|} : q \in Q;\ \omega \geq 0 \right\} - 1$$

for an uncertain system $M(P_0(s,q),r(s),Q)$.

<u>Step 3</u> (Inductive Step): Given $\Delta_i(s,q)$ and $D_{c,i}(s)$, select $\varepsilon_i > 0$ such that

$$\Delta_{i+1}(s,q) \triangleq \Delta_i(s,q) + \varepsilon_i s^i D(s,q) \tag{4.35}$$

is Hurwitz invariant,

$$D_{c,i+1}(s) \triangleq D_{c,i}(s) + \varepsilon_i s^i \tag{4.36}$$

is Hurwitz (when a <u>stable</u> controller is required) and

$$\frac{|\Delta_i(j\omega,q)|}{|\Delta_{i+1}(j\omega,q)|} \leq \frac{1 + \delta/2 + \delta/4 + \ldots + \delta/2^{i+1}}{1 + \delta/2 + \delta/4 + \ldots + \delta/2^i} \tag{4.37}$$

for all $q \in Q$ and $\omega \geq 0$. Assumptions 4.1-4.4 guarantee the success of this procedure for $i = 0,1,2,\ldots,l$.

<u>Step 4</u> (Termination): The denominator of the compensator is given by

$D_c(s) \Delta D_{c,l+1}(s)$.

Remark: With the model proposed in the last section, one may think of an alternative design procedure for a robust stabilizer: take a plant $P_0(s)$ having the highest relative degree from $P_0(s,q)$ as the nominal plant. Design a controller to stabilize $P_0(s)$ with its high frequency uncertainty and then increase the control gain to stabilize the rest of the systems in the family. The following example shows that this method cannot lead to a desired robust stabilizer, even for an uncertain system without high frequency unstructured uncertainty. For example, consider a family of systems:

$$P_0(s,1) = \frac{100}{s^3};$$

$$P_0(s,2) = \frac{s^2 + 2s + 1}{s^4 - 20s^3}.$$

Obviously, the compensator

$$C(s) = \frac{s^2 + 2s + 1}{s^2 + 10s + 50}$$

stabilizes the system $P(s,1)$. However, it is easy to verify that for any $k \in R$, the controller $C_1(s) = kC(s)$ cannot stabilize the system $P(s,2)$. However, with the method given in [60], one may find a robust stabilizer as follows:

$$C(s) = \frac{2 \times 10^5(s^2 + 2s + 1)}{s^2 + 8000s + 50000}$$

Illustrative Examples

In this section, two examples are provided to illustrate the construction of a robust compensator for the combined uncertainty case.

Example 4.1

An uncertain system $M(P_0(s,q),r(s),Q)$, as in (4.31), is characterized by $Q = \{1,2,3\}$ and

$$P_0(s,1) = \frac{1}{s^2 - 2s - 2};$$

$$P_0(s,2) = \frac{1}{s - 1};$$

$$P_0(s,3) = \frac{s^2 + 2s + 1}{s^3 - 2s^2};$$

$$r(s) = \frac{0.1}{s + 1}.$$

It is easy to verify that Assumptions 4.1-4.4 are satisfied; hence, the system is stabilizable. A robust compensator $C(s)$ can be constructed as follows:

Step 1: We now require $k \geq \max\{2,1,1\} - 1 = 1$. Taking $k = 1$ and $l = 1$, the compensator has the form

$$C(s) = \frac{\lambda_1 s + \lambda_0}{\varepsilon_1 s + \varepsilon_0}; \quad \lambda_1 > 0.$$

Next, we choose $\lambda_1 s + \lambda_0 = s + 1$, i.e., $N_c(s) = s + 1$.

Step 2: We have

$\Delta_0(s,1) = s + 1,$

$\Delta_0(s,2) = s + 1,$

$\Delta_0(s,3) = (s + 1)(s^2 + 2s + 1)$, and

$\delta = \inf\{1/|r(j\omega)|: \omega \geq 0\} - 1 = 9.$

Steps 3-4: To select ε_0, consider

$\Delta_1(s,1) = \varepsilon_0(s^2 - 2s - 2) + \Delta_0(s,1),$

$\Delta_1(s,2) = \varepsilon_0(s - 1) + \Delta_0(s,2),$

$\Delta_1(s,3) = \varepsilon_0(s^3 - 2s^2) + \Delta_0(s,3)$

and

$$\frac{|\Delta_0(j\omega,q)|}{|\Delta_1(j\omega,q)|} \le 1 + \delta/2 = 5.5 \text{ for } q \in \{1,2,3\}.$$

In view of all three cases, we can take $\varepsilon_0 = 0.1$ to assure Hurwitz invariance of $\Delta_1(s,q)$ and satisfaction of the inequality above. Now, we again consider three cases of

$$\Delta_2(s,1) = \varepsilon_1 s(s^2 - 2s - 2) + \Delta_1(s,1),$$

$$\Delta_2(s,2) = \varepsilon_1 s(s - 1) + \Delta_1(s,2),$$

$$\Delta_2(s,3) = \varepsilon_1 s(s^3 - 2s^2) + \Delta_1(s,3),$$

and

$$\frac{|\Delta_1(j\omega,q)|}{|\Delta_2(j\omega,q)|} \le \frac{1 + \delta/2 + \delta/4}{1 + \delta/2} = 1.409 \text{ for all } q \in \{1,2,3\}.$$

we can assure Hurwitz invariance of $\Delta_2(s,q)$ and satisfaction of the inequality above by taking $\varepsilon_1 = 0.005$. Hence, a stabilizing compensator is given by

$$C(s) = \frac{s + 1}{0.005s + 0.1} = \frac{200(s + 1)}{s + 20}.$$

<u>Example 4.2</u>

An uncertain system $M(P_0(s,q),r(s),Q)$, as in (4.34), is characterized by

$$P_0(s,q) = \frac{q_1}{s + q_2}\ ; \quad q_1 \in [1,2];\ q_2 \in [-1,1],$$

$$r(s) = \frac{0.5}{s + 1}.$$

In this example, the uncertain system has continuous parameter variation. It is easy to verify that Assumptions 4.1-4.4 are satisfied, hence, the system is stabilizable. A robust compensator $C(s)$ can be constructed as follows:

<u>Step 1</u>: We now require $k \ge 0$. Taking $k = 0$ and $l = 1$, the compensator has the form

$$C(s) = \frac{\lambda_0}{\varepsilon_1 s + \varepsilon_0} \; ; \quad \lambda_0 > 0.$$

Next, we choose $\lambda_0 = 1$, i.e., $N_c(s) = 1$.

<u>Step 2</u>: We have

$$\Delta_0(s,q) = q_1$$

$$\delta = \inf\{1/|r(j\omega)| : \omega \geq 0\} - 1 = 1.$$

Steps 3-4: Now

$$\Delta_1(s,q) = \varepsilon_0(s + q_2) + q_1 = \varepsilon_0 s + (\varepsilon_0 q_2 + q_1).$$

$$\frac{|\Delta_0(j\omega,q)|}{|\Delta_1(j\omega,q)|} \leq +\delta/2;$$

or

$$\frac{q_1^2}{\varepsilon_0^2\omega^2 + (q_1 + \varepsilon_0 q_2)^2} \leq 9/4$$

With the given bounds of q_i, we require $\varepsilon_0 < 1$ for Hurwitz invariance of $\Delta_1(s,q)$ and $\varepsilon_0 < 1/3$ for satisfying the inequality above. Hence, we take $\varepsilon_0 = 0.2$. A straightforward computation results in

$$\Delta_2(s,q) = \varepsilon_1 s^2 + \varepsilon_1 q_2 s + 0.2s + 0.2q_2 + q_1$$

and we require

$$\frac{|\Delta_1(j\omega,q)|}{|\Delta_2(j\omega,q)|} \leq \frac{1 + \delta/2 + \delta/4}{1 + \delta/2}$$

or

$$\frac{0.04\omega^2 + (q_1 + 0.2q_2)^2}{(\varepsilon_1 q_2 + 0.2)^2 + (0.2q_2 + q_1 - \varepsilon_1\omega^2)^2} \leq 49/36.$$

With the given bounds of q_i, we can assure Hurwitz invariance of $\Delta_2(s,q)$ with $\varepsilon_1 < 0.2$ and satisfaction of the inequality above with $\varepsilon_1 < 0.0026$. Hence, we can take $\varepsilon_1 = 0.002$ and the stabilizing compensator is now given by

$$C(s) = \frac{1}{0.002s + 0.2} = \frac{500}{s + 100}.$$

In this section, we proposed a new design model for uncertain systems containing both parameter variation and unstructured uncertainty. Sufficient conditions are provided under which robust stabilizability of SISO uncertain systems is guaranteed. Furthermore, the robust compensator can be constructed to be strictly proper and stable; its numerator can be any stable polynomial having degree k given in Step 1 of Procedure 4.1. Its denominator has coefficients ε_i which can be computed one at a time using the recursion given in Step 3 of Procedure 4.1.

Some Comments on Computational Requirements:

The proposed design procedure, of course, demands a fairly good amount of computation, both in terms of labor and complexity. Firstly, one needs an algorithm to test the Hurwitz invariance of uncertain polynomials. The Kharitonov test and related improvements as discussed in section II would be of assistance in this regard. Next, one needs to evaluate the inequality of (4.37) for all frequencies $\omega \geq 0$. This step is more laborious than complex. One attractive or comforting feature of this design procedure is that it is recursive in nature. Thus at any given step in the computational procedure, there is only one parameter to be determined, which significantly simplifies the computational burden.

V. CONCLUSIONS AND RECOMMENDATIONS FOR FUTURE RESEARCH

5.1 THE WORK IN RETROSPECT

The main theme of the described research under the present contract has been to analyze and synthesize controllers for robust stability for linear time invariant systems with structured parametric uncertainty, both from a time domain and a frequency domain perspective. First the aspect of analysis for stability robustness is considered. A useful review of currently available methods of analysis (namely, "polynomial" and "matrix" methods) is given. Then, an algorithm is presented to reduce the conservatism of Kharitonov test for testing the Hurwitz invariance of uncertain polynomials with dependent coefficients. Also, for systems described by state space models, improved upper bounds for robust stability are obtained by further exploiting the functional dependence of the elements of the perturbation on the uncertain parameters. The contents of this section cover the task described as "task 3" in the original proposal.

Next, the aspect of synthesis of controllers is addressed. A robust control design method is presented for systems described by state space representation such that it maximizes (in some sense) the stability robustness bound for a given structure of the uncertainty, taking into account the control effort constraints. This algorithm adequately answers the question raised in "task 1" of the proposal. Then, in addition, another design method from frequency domain point of view is also presented such that a single

compensator simultaneously stabilizes a class of non-minimum phase systems. The conditions under which the existence of the controller is guaranteed are given along with an algorithm to construct the compensator.

Then the aspect of combined structured and unstructured uncertainty is considered. First in the time domain framework, specifically for large space structure models, a problem formulation is presented in which the unmodeled dynamics are characterized as an additive perturbation and control gain bounds are derived such that the closed loop system remains stable even in the presence of both control and observation spillovers. The design algorithm is then extended to account for parameter variations in the lower order control design model. In other words, the unstructured and structured uncertainty cases were obtained in sequence of one another rather than being simultaneously present. However, the design algorithm can be extended to the case when both types of errors are present simultaneously. This section of the report covers the task defined as "task 2" of the original proposal.

Finally, the aspect of simultaneous presence of structured and unstructured uncertainty in the system is addressed from frequency domain viewpoint. An algorithm is presented to design a single (reduced order) compensator that guarantees robust stabilization under the combined presence of these two types of uncertainties. The conditions for the existence of the controller and its construction are also given in detail. This result covers the task raised as "task 4" of the proposal.

The publications listed as References [87]-[97] are the result of this research study.

As it normally occurs, another result of this study is that many interesting research topics have surfaced for further investigation. These are described in the next section.

5.2 TOPICS FOR FURTHER RESEARCH WHICH NEED CONTINUED SUPPORT

(1) The foremost area of research would be to extend the frequency domain design methodology for the combined structured and unstructured uncertainty problem by relaxing some of the assumptions made with respect to the minimum phase condition, and to enlarge the class of unstructured uncertainty profiles considered in the synthesis of the compensator for robust stabilization.

(2) Another topic for further research would be the incorporation, in addition to the robust stabilization requirement, of the requirement of performance robustness in the problem formulation. Some specific constraints to be considered could be disturbance rejection and bounded control effort.

(3) An interesting area of research would be to develop the state space conterpart to the frequency domain design methodology given for the combined uncertainty problem.

(4) One aspect that needs attention is the extension of the proposed frequency domain methodology under combined uncertainty problem to Multiple Input Multiple Output (MIMO) systems. This is definitely not a straightforward extension of the Single Input Single Output (SISO) case.

(5) It is also important to explore the possible relationship of different methodologies available, such as structured singular value design, multivariable stability margin design, simultaneous stabilization design using polynomial theory, and the H_∞ control design, since all of these approaches seem to solve the same problem.

(6) Of course, it is always beneficial to apply the developed methodologies to specific applications, such as aircraft/spacecraft control, control of flexible structures, and robotics.

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APPENDIX A: PROOF OF THEOREM 2.1

Necessity is trivially proved by letting $k_1(q) = k_2(q) = 1$.

(Sufficiency): When n is even, let

$$h(s^2,q) = \alpha_0(q)s^n + \alpha_2(q)s^{n-2} + \alpha_4(q)s^{n-4} + \ldots + \alpha_n(q)$$

and

$$sg(s^2,q) = \alpha_1(q)s^{n-1} + \alpha_3(q)s^{n-3} + \alpha_5(q)s^{n-5} + \ldots + \alpha_{n-1}(q)s.$$

Then

$$\begin{aligned} f'(s,q) &= k_1(q)h(s^2,q) + k_2(q)sg(s^2,q) \\ &= k_1(q)h(z,q) + sk_2(q)g(z,q). \end{aligned}$$

It follows from the Hermite-Bieler theorem that if $f'(s,q)$ is Hurwitz invariant over Q, then $k_1(q)h(z,q)$ and $k_2(q)g(z,q)$ form a positive pair for every $q \in Q$, i.e., the zeros of the polynomials $k_1(q)h(z,q)$ and $k_2(q)g(z,q)$ must be distinct, real, negative, and interlaced as follows:

$$u_1 < v_1 < u_2 < v_2 < \ldots < v_{m-1} < u_m < 0$$

where $m = n/2$, u_i are zeros of $k_1(q)h(z,q)$ and v_i are zeros of $k_2(q)g(z,q)$. However, $k_1(q)h(z,q)$ and $h(z,q)$ have the same zeros. Hence, $h(z,q)$ and $g(z,q)$ also form a positive pair for every $q \in Q$. Thus, the Hermite-Bieler theorem guarantees that

$$f(s,q) = h(z,q) + sg(z,q)$$

is also Hurwitz invariant over Q.

When n is odd, a similar proof leads to the same conclusion.

APPENDIX B: PROOF OF THEOREM 2.3

By assumption the four bounding polynomials of f(s,q) are Hurwitz. It follows from the Kharitonov theorem that every polynomial

$$f(s) = \alpha_0 s^n + \alpha_1 s^{n-1} + \ldots + \alpha_n, \quad \alpha_i \in [a_i, b_i]$$

is Hurwitz. Since

$$a_i \leq a_i', \qquad b_i' \leq b_i,$$

then every polynomial

$$f(s) = \alpha_0 s^n + \alpha_1 s^{n-1} + \ldots + \alpha_n, \quad \alpha_i \in [a_i', b_i']$$

is also Hurwitz. Consequently, the four bounding polynomials of f'(s,q) are Hurwitz.

APPENDIX C: PROOF OF THEOREM 2.4

It is known that the perturbed system

$$\dot{x}(t) = (A_0 + E)x(t), \tag{C.1}$$

where A_0 is an asymptotically stable matrix, is asymptotically stable if

$$\sup_{\omega \geq 0} \rho[(j\omega I - A_0)^{-1}E] < 1 \tag{C.2}$$

Let $(j\omega I - A_0)^{-1} = M(\omega)$. Now with E given by

$$E = \sum_{i=1}^{r} \beta_i E_i, \tag{C.3}$$

the perturbed system (F.1) is asymptotically stable if

$$\sup_{\omega \geq 0} \rho \left[M(\omega) \left(\sum_{i=1}^{r} \beta_i E_i \right) \right] < 1 \tag{C.4}$$

But

$$\max_j |\beta_j| \sup_{\omega \geq 0} \rho \left[\sum_{i=1}^{r} |M(\omega)E| \right] \geq \sup_{\omega \geq 0} \rho \left(\sum_{i=1}^{r} |\beta_i M(\omega) E_i| \right)$$

$$\geq \sup_{\omega \geq 0} \rho \left(\sum_{i=1}^{r} \beta_i M(\omega) E_i \right)$$

$$\geq \sup_{\omega \geq 0} \rho \left[M(\omega) \left(\sum_{i=1}^{r} \beta_i E_i \right) \right]$$

The satisfaction of condition (2.12a) implies the satisfaction of (C.4). Hence, the perturbed system is asymptotically stable.

For $r = 1$, we see that

$$|\beta_1| \sup_{\omega \geq 0} \rho [M(\omega)E_1)] \geq \sup_{\omega \geq 0} \rho [M(\omega)(\beta_1 E_1)]$$

Hence (2.12b) implies (C.4), leading to the result.

APPENDIX D: PROOF OF LEMMAS 3.3 AND 3.4

By the hypothesis of Lemma 3.3, the index set Q can be divided into a finite number of subsets $Q_{nn'}$:

$$Q_{nn'} = \{q \in Q: \quad d(q) = n; \quad d'(q) = n'\}.$$

In each subset $Q_{nn'}$, it follows from Lemma 3.1 that there exists an $\varepsilon_{nn'} > 0$ such that for any $\varepsilon \in (0, \varepsilon_{nn'})$,

$$f_\varepsilon(s,q) = f(s,q) + \varepsilon f'(s,q)$$

is also PHI over $Q_{nn'}$. Hence, the conclusion of Lemma 3.3 immediately follows by taking

$$\varepsilon^* = \min_{n,n'} \{\varepsilon_{nn'}\}.$$

A similar proof applies to Lemma 3.4.

APPENDIX E: PROOF OF THEOREM 3.2

By assumption that $N_1(s,q)$ is Hurwitz invariant, it follows from Theorem 3.1 that there exists a $C_0(s) = N_c(s)/D_{c,0}(s)$ such that

$$\Delta_0(s,q) = N_c(s)N_1(s,q) + D_{c,0}(s)D(s,q)$$

is PHI over Q.

Now, applying Lemma 3.4, there exists an $\varepsilon_0 > 0$ such that

$$\Delta_1(s,q) = s\Delta_0(s,q) + \varepsilon_0 D_{c,0}(s)D(s,q)$$

is PHI over Q. Similarly, there exists an $\varepsilon_1 > 0$ such that

$$\Delta_2(s,q) = s\Delta_1(s,q) + \varepsilon_1 D_{c,0}(s)D(s,q)$$

is PHI over Q, and there exists an $\varepsilon_2 > 0$ such that

$$\begin{aligned}\Delta_3(s,q) &= s\Delta_2(s,q) + \varepsilon_2 D_{c,0}(s)D(s,q)\\ &= s^3\Delta_0(s,q) + (\varepsilon_0 s^2 + \varepsilon_1 s + \varepsilon_2)D_{c,0}(s)D(s,q)\end{aligned}$$

is PHI over Q and $D_{c,3}(s) = \varepsilon_0 s^2 + \varepsilon_1 s + \varepsilon_2$ is Hurwitz.

In the next step, it follows from Lemma 3.4 that there exists an $\varepsilon^* > 0$ such that for any $\varepsilon \in (0,\varepsilon^*)$,

$$f_\varepsilon(s,q) = s\Delta_3(s) + D_{c,0}(s)D(s,q)$$

is PHI over Q. Also, there exists an $\varepsilon^{**} > 0$ such that for every $\varepsilon \in (0,\varepsilon^{**})$,

$$f'_\varepsilon(s) = sD_{c,3}(s) + \varepsilon$$

is Hurwitz. Hence, we can take $0 < \varepsilon_3 \quad \min(\varepsilon^*,\varepsilon^{**})$ to guarantee that

$$\Delta_4(s,q) = s\Delta_3(s) + \varepsilon_3 D_{c,0}(s)D(s,q)$$

is PHI and

$$D_{c,4}(s) = sD_{c,3}(s) + \varepsilon_3 = \varepsilon_0 s^3 + \varepsilon_1 s^2 + \varepsilon_2 s + \varepsilon_3$$

is Hurwitz.

Repeating the above procedure for λ-1 times, we complete the proof.

APPENDIX F: PROOF OF THEOREM 3.3

By applying Lemma 3.4, we take $f(s,q) = D_1(s,q)$ and $f'(s,q) = N(s,q)$ and follow the same procedure as in Appendix E to construct

$$N_c(s) = \delta_0 s^{\lambda-1} + \delta_1 s^{\lambda-2} + \ldots + \delta_{\lambda-1}$$

such that

$$\Delta_1'(s,q) = N_c(s)N(s,q) + s^{\lambda}D_1(s,q)$$

is PHI. Then we apply Lemma 3.3 by taking $f(s,q) = \Delta_1'(s,q)$ and $f'(s,q) = sD(s,q) = s^{\lambda+1}D_1(s,q)$. Hence, there exists an $\epsilon_1 > 0$ such that

$$\Delta_2'(s,q) = \Delta_1'(s,q) + \epsilon_1 sD(s,q) = N_c(s)N(s,q) + (1 + \epsilon_1 s)D(s,q)$$

is PHI over Q. Similarly, there exists an $\epsilon_2 > 0$ such that

$$\Delta_3'(s,q) = \Delta_2'(s,q) + \epsilon_2 s^2 D(s,q)$$

is PHI over Q and $D_{c,2}(s) = 1 + \epsilon_1 s + \epsilon_2 s^2$ is Hurwitz.

In the next step, it follows from Lemma 3.4 that there exists an $\epsilon^* > 0$ such that for any $\epsilon \in (0,\epsilon^*)$,

$$f_\epsilon(s,q) = \Delta_3'(s) + \epsilon s^3 D(s,q)$$

is PHI over Q. Also there exists an $\epsilon^{**} > 0$ such that for every $\epsilon \in (0,\epsilon^{**})$,

$$f_\epsilon'(s) = D_{c,2}(s) + \epsilon s^3$$

is Hurwitz. Hence, we can take $0 < \epsilon_3 < \min(\epsilon^*,\epsilon^{**})$ to guarantee that

$$\Delta_4'(s,q) = \Delta_3'(s) + \epsilon_3 s^3 D(s,q)$$

is PHI and

$$D_{c,3}(s) = D_{c,2}(s) + \epsilon_3 s^3 = 1 + \epsilon_1 s + \epsilon_2 s^2 + \epsilon_3 s^3$$

is Hurwitz.

Repeating the above procedure for $l \geq \lambda-1$ times, we complete the proof.

APPENDIX G: PROOF OF THEOREM 3.4

By assumption that $P_1(s,q)$ is simultaneously stabilizable, there exists a compensator $C_1(s) = N_{c,1}(s)/D_{c,1}(s)$ such that

$$\Delta(s,q) = D_{c,1}(s)D_1(s,q) + N_{c,1}(s)N_1(s,q)$$

is Hurwitz invariant over Q. Now, construct a compensator

$$C(s) = \frac{C_1(s)}{1 - C_1(s)P_0(s)} = \frac{N_{c,1}(s)D_0(s)}{D_0(s)D_{c,1}(s) - N_0(s)N_{c,1}(s)}$$

and the closed loop system

$$P^*(s,q) = \frac{P(s,q)}{1 + C(s)P(s,q)} = \frac{N(s,q)D_c(s)}{D(s,q)D_c(s) + N(s,q)N_c(s)}$$

$$= \frac{N(s,q)D_c(s)}{D_0^2(s)D_1(s,q)D_{c,1}(s) + N_1(s,q)N_{c,1}(s)}$$

Since $D_0(s)$ is Hurwitz and $D_1(s,q)D_{c,1}(s) + N_1(s,q)N_{c,1}(s)$ is Hurwitz invariant over Q, the closed loop system is robust stable.

APPENDIX H: PROOF OF LEMMA 4.1

Knowing that $\beta_0(q) = \alpha_0^2(q)$ is a continuous function in the compact set Q and $\alpha_0(q) > 0$ for all $q \in Q$, it immediately follows that

$$\min \{\beta_0(q): q \in Q\} = b_0 > 0.$$

Since f(s,q) is PHI, we have

$$g(\omega^2,q) > 0$$

for every $q \in Q$ and $\omega \geq 0$. The standardness of f(s,q) guarantees that $\beta_0(q),\beta_1(q),\ldots,\beta_n(q)$ are all bounded and $\min \{\beta_0(q):q \in Q\} = b_0 > 0$, so for any $k_1 > 0$, there exists an $\omega_k > 0$ such that for all $\omega > \omega_k$ and $q \in Q$,

$$g(\omega^2,q) > k_1.$$

Now $g(\omega^2,q)$ is a continuous function in a compact set $\{[0,\omega_k],Q\}$, so it follows that

$$\min \{g(\omega^2,q): q \in Q,\ \omega \in [0,\omega_k]\} = k_2 > 0.$$

Consequently,

$$\min \{g(\omega^2,q): q \in Q,\ \omega \geq 0\} = \min \{k_1,k_2\} = k > 0.$$

APPENDIX I: PROOF OF LEMMA 4.2

By assumption that $f(s,q)$ and $f'(s,q)$ are standard, the index set Q can be divided into a finite number of compact subsets $Q_{nn'}$, where

$$Q_{nn'} = \{q \in Q: d(q) = n,\ d'(q) = n'\}.$$

First, we prove that the lemma is true when $q \in Q_{nn'}$; i.e., given any $a > 1$, there exists an $\varepsilon_{nn'} > 0$ such that for any $\varepsilon \in (0,\varepsilon_{nn'})$,

$$f_\varepsilon(s,q) = f(s,q) + \varepsilon f'(s,q)$$

is also PHI over $Q_{nn'}$ and

$$\frac{|f(j\omega,q)|}{|f_\varepsilon(j\omega,q)|} < a$$

for all $q \in Q_{nn'}$ and $\omega \geq 0$. Then, by letting

$$\varepsilon^* = \min\{\varepsilon_{nn'}: \forall\ n,n'\},$$

the conclusion of Lemma 4.2 directly follows.

It follows from Lemma 4.5 that there exists an $\varepsilon_1 > 0$ such that for any $\varepsilon \in (0,\varepsilon_1)$,

$$f_\varepsilon(s,q) = f(s,q) + \varepsilon f'(s,q)$$

is also PHI over $Q_{nn'}$.

We now claim that for any $a > 1$, there exists an $\varepsilon_2 > 0$ such that for any $\varepsilon \in (0,\varepsilon_2)$,

$$\frac{|f(j\omega,q)|}{|f_\varepsilon(j\omega,q)|} < a$$

for all $q \in Q_{nn'}$ and $\omega \geq 0$. Once the claim is proved, the conclusion immediately follows by letting $\varepsilon_{nn'} = \min\{\varepsilon_1,\varepsilon_2\}$.

To prove the claim, it is equivalent to show that there exists an $\varepsilon_2 > 0$ such that for any $\varepsilon \in (0,\varepsilon_2)$,

$$\frac{|f(j\omega,q)|^2}{|f_\varepsilon(j\omega,q)|^2} < 1 + \delta_1$$

for all $q \in Q_{nn'}$ and $\omega \geq 0$ where $\delta_1 = a^2 - 1 > 0$; or, equivalently,

$$(1 + \delta_1)|f_\varepsilon(j\omega,q)|^2 - |f(j\omega,q)|^2 > 0.$$

<u>Case 1</u>: $n' = n + 1$

A straightforward computation yields

$$\begin{aligned}
&(1 + \delta_1)|f_\varepsilon(j\omega,q)|^2 - |f(j\omega,q)|^2 \\
&\quad = (1 + \delta_1)\varepsilon^2\alpha_0'^2(q)\omega^{2n+2} + h_1(\omega^2,q,\varepsilon,\delta_1) + \delta_1|f(j\omega,q)|^2 \\
&\quad = (1 + \delta_1)\varepsilon^2\alpha_0'^2(q)\omega^{2n+2} + h_1(\omega^2,q,\varepsilon,\delta_1) + \delta_1 g(\omega^2,q) \\
&\quad \Delta\ (1 + \delta_1)\varepsilon^2\alpha_0'^2(q)\omega^{2n+2} + h(\omega^2,q,\varepsilon,\delta_1)
\end{aligned}$$

where $\alpha_0'^2(q)$ is the leading coefficient of $f'(s,q)$ and

$$h_1(\omega^2,q,\varepsilon,\delta_1) = \varepsilon(\gamma_0(q,\varepsilon,\delta_1)\omega^{2n} + \gamma_1(q,\varepsilon,\delta_1)\omega^{2n-2} + \ldots + \gamma_n(q,\varepsilon,\delta_1))$$

$$g(\omega^2,q)\ \Delta\ |f(j\omega,q)|^2 = \beta_0(q)\omega^{2n} + \beta_1(q)\omega^{2n-2} + \ldots + \beta_n(q).$$

The standardness of $f(s,q)$ and $f'(s,q)$ guarantees that for given δ_1 when $\varepsilon \to 0$, $\varepsilon\gamma_i(q,\varepsilon,\delta_1)$ <u>uniformly</u> converges to zero in the compace set $Q_{nn'}$ for all $i = 0,1,\ldots,n$. By Lemma 4.1,

$$\min\{\beta_0(q): q \in Q_{nn'}\} = b_0 > 0.$$

So, there exists an $\varepsilon_3 > 0$ such that for any $\varepsilon \in (0,\varepsilon_3)$,

$$\min\{\delta_1\beta_0(q) + \varepsilon\gamma_0(q,\varepsilon,\delta_1): q \in Q_{nn'}\} > 0.5\delta_1 b_0 > 0.$$

Consequently, for any $M_1 > 0$, there exists a $\omega_m > 0$ such that for any $\omega > \omega_m$ and $q \in Q_{nn'}$,

$$h(\omega^2,q,\varepsilon,\delta_1) > M_1.$$

For given δ_1, $h(\omega^2,q,\varepsilon,\delta_1)$ is a continuous function of ε in the compact set $\{[0,\omega_m],Q_{nn'}\}$, it follows that when $\varepsilon \to 0$, $h(\omega^2,q,\varepsilon,\delta_1)$ <u>uniformly</u> converges to $\delta_1 g(\omega^2,q)$. Hence, there exists an $\varepsilon_4 > 0$ such that for any $\varepsilon \in (0,\varepsilon_4)$,

$$\min\{h(\omega^2,q,\varepsilon,\delta_1):\ q \in Q_{nn'},\ \omega \in [0,\omega_m]\} > 0.5\delta_1 k > 0$$

where $k = \min\{g(\omega_2,q):\ q \in Q_{nn'},\ \omega \geq 0\} > 0$ by Lemma 4.6. Now take $\varepsilon_2 = \min\{\varepsilon_3,\varepsilon_4\} > 0$ and $M = \min\{M_1, 0.5\delta_1 k\}$. Then, for any $\varepsilon \in (0,\varepsilon_2)$,

$$\min\{h(\varepsilon^2,q,\varepsilon,\delta_1):\ q \in Q_{nn'},\ \omega \geq 0\} > M > 0.$$

Since $(1+\delta_1)\varepsilon^2\alpha_0'^2(q)\omega^{2n+2} \geq 0$, we conclude that for any $\varepsilon \in (0,\varepsilon_2)$,

$$(1+\delta_1)|f_\varepsilon(j\omega,q)|^2 - |f(j\omega,q)|^2 > 0$$

for all $q \in Q_{nn'}$ and $\omega \geq 0$.

<u>Case 2</u>: $n' < n+1$

In this case, the term ω^{2n+2} no longer exists and the proof above remains valid.

APPENDIX J: PROOF OF THEOREM 4.2

Consider an uncertain system $A(P_0(s,q),r(s),Q)$ satisfying Assumptions 4.1-4.4. Let $C(s) \triangleq N_c(s)/D_c(s)$. It follows from Corollary 4.1 that we need to show

$$\Delta(s,q) \triangleq D_c(s)D(s,q) + N_c(s)N(s,q) \tag{J.1}$$

is Hurwitz invariant over Q and

$$\frac{|N_c(j\omega)N(j\omega,q)|\ \ |r(j\omega)|}{|D_c(j\omega)D(j\omega,q) + N_c(j\omega)N(j\omega,q)|\ \ |P_0(j\omega,q)|} \leq 1$$

for all $q \in Q$ and $\omega \geq 0$. By Assumption 4.4, we have

$$\sup\left\{ \frac{|r(j\omega)|}{|P_0(j\omega,q)|} : \quad q \in Q;\ \omega \geq 0 \right\} < 1.$$

Consequently, there exists a $\delta > 0$ such that

$$\frac{|r(j\omega)|}{|P_0(j\omega,q)|} < 1/(1 + \delta).$$

for all $q \in Q$ and $\omega \geq 0$. Hence, we only need to show that

$$\frac{|N_c(j\omega)N(j\omega,q)|}{|D_c(j\omega)D(j\omega,q) + N_c(j\omega)N(j\omega,q)|} \leq 1 + \delta \tag{J.2}$$

for all $q \in Q$ and $\omega \geq 0$.

To construct a compensator $C(s)$ satisfying (J.1) and (J.2), begin by choosing

$$N_c(s) \triangleq \sum_{i=0}^{k} \lambda_i s^i$$

to be any Hurwitz polynomial with $\lambda_k > 0$ when $N(s,q)$ is PHI, or $\lambda_k < 0$ when $N(s,q)$ is negative Hurwitz invariant (NHI) and of degree

$$k \geq \max\{d'(q) - d(q): \ q \in Q\} - 1.$$

Note that if $\max\{d'(q) - d(q): q \in Q\} = 0$, we simply take $k \geq 0$.

Recall that $D(s,q)$ is a standard uncertain polynomial and $N(s,q)$ is a standard PHI or NHI polynomial. It follows that $\Delta_0(s,q) \Delta N_c(s)N(s,q)$ is also a standard PHI polynomial. Take

$$f(s,q) = \Delta_0(s,q) \text{ and}$$

$$f'(s,q) = D(s,q).$$

It is easy to verify that all hypotheses of Lemma 4.2 are satisfied. Hence, we can select a sufficiently small $\epsilon_0 > 0$ such that

$$\Delta_1(s,q) \Delta \Delta_0(s,q) + \epsilon_0 D(s,q)$$

is a standard PHI polynomial and

$$\frac{|\Delta_0(j\omega,q)|}{|\Delta_1(j\omega,q)|} \leq 1 + \delta/2$$

for all $q \in Q$ and $\omega \geq 0$. The proof is continued by repeated application of Lemma 4.2; i.e., take

$$f(s,q) = \Delta_1(s,q)$$

$$f'(s,q) = sD(s,q)$$

and again, the requirements of the lemma are satisfied. Hence, we can choose $\epsilon_1 > 0$ small enough that

$$\Delta_2(s,q) \Delta \Delta_0(s,q) + (\epsilon_0 + \epsilon_1 s)D(s,q)$$

is a standard PHI polynomial and

$$\frac{|\Delta_1(j\omega,q)|}{|\Delta_2(j\omega,q)|} \leq \frac{1 + \delta/2 + \delta/4}{1 + \delta/2}$$

for all $q \in Q$ and $\omega \geq 0$. Continuing in this manner, we now apply Lemma 4.2 with $f(s,q) = \Delta_2(s,q)$ and $f'(s,q) = s^2D(s,q)$, and can generate $\epsilon_2 > 0$ such that

$$\Delta_3(s,q) \triangleq \Delta_2(s,q) + \varepsilon_2 f'(s,q) = \Delta_0(s,q) + (\varepsilon_0 + \varepsilon_1 s + \varepsilon_2 s^2)D(s,q)$$

is again a standard PHI polynomial and

$$\frac{|\Delta_2(j\omega,q)|}{|\Delta_3(j\omega,q)|} \leq \frac{1 + \delta/2 + \delta/4 + \delta/8}{1 + \delta/2 + \delta/4}$$

for all $q \in Q$ and $\omega \geq 0$ <u>and</u> that

$$D_{c,4}(s) \triangleq \varepsilon_0 + \varepsilon_1 s + \varepsilon_2 s^2 + \varepsilon_3 s^3$$

is Hurwitz. By continuing step by step in this manner, we can generate a Hurwitz polynomial

$$D_c(s) \triangleq D_{c,\ell+1}(s) = \sum_{i=0}^{\ell} \varepsilon_i s^i$$

such that $\ell \geq k$ and $\Delta(s,q)$ is PHI and

$$\frac{|\Delta_{\ell-1}(j\omega,q)|}{|\Delta_\ell(j\omega,q)|} \leq \frac{1 + \delta/2 + \delta/4 + \ldots + \delta/2^\ell}{1 + \delta/2 + \delta/4 + \ldots + \delta/2^{\ell-1}}$$

for all $q \in Q$ and $\omega \geq 0$. Consequently,

$$\frac{|N_c(j\omega)N(j\omega,q)|}{|D_c(j\omega)D(j\omega,q) + N_c(j\omega)N(j\omega,q)|} = \frac{|\Delta_0(j\omega,q)|}{|\Delta_1(j\omega,q)|}$$

$$= \frac{|\Delta_0(j\omega,q)|}{|\Delta_1(j\omega,q)|} \frac{|\Delta_1(j\omega,q)|}{|\Delta_2(j\omega,q)|} \cdots \frac{|\Delta_{\ell-1}(j\omega,q)|}{|\Delta_\ell(j\omega,q)|}$$

$$\leq 1 + \delta/2 + \delta/4 + \ldots + \delta/2^\ell < 1 + \delta$$

for all $q \in Q$ and $\omega \geq 0$.

A similar proof applies to an uncertain system $M(P_0(s,q),r(s),Q)$.